DYNAMIC MODELLING OF SWITCHING SYSTEMS

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Abstract. A generalized approach for obtaining a small-signal model of linear switching systems is described. Starting from a piece-wise constant linear description of the system, a concise, discrete-time model is obtained using the augmented state vector method. This model together with a generalized description of the transition conditions between the modes is used to obtain the cyclic state of the system. Linearization of both the model and the transition conditions yields a linear small-signal model, which is valid in the vicinity of an operating point. Due to the augmented state form of the system, this linearized model can easily be calculated. This method for obtaining a small-signal model is applicable to a large class of systems. As an example, the advantages of the method are shown in an experiment with a DC-motor drive.

Keywords. Exact modelling, linearization, control, PWM, state vector augmentation

INTRODUCTION

Modelling of power electronic converters has been subject of intense research activity in recent times [1]-[6]. These converters contain power sources and dynamic elements (capacitors and inductors) interconnected by resistors and switches. With the switches the configuration of the network can be altered. Therefore, the model of the system is not constant, but changes discontinuously at the switching instants. Between the switching moments, however, the system parameters are considered to be constant. Thus, a complete model of the system consists of a set of models, each one valid in consecutive intervals of time.

For analysing the behaviour of the system the differential equations governing these models have to be solved, and the solutions combined. Using the augmented state vector method yields concise models which can easily be manipulated [1]. Further, models describing the dynamic behaviour of the network are obtained via linearization. Due to the augmented state form of the system, the resulting expressions can easily be calculated. With these models the behaviour of the controlled system can be evaluated for all operating points. The merits of the proposed modelling technique are shown in an experiment with a DC-motor drive.

SYSTEM FORMULATION

In this section a definition of the class of systems under study will be given. Switching systems have two main characteristics. First, switching systems display a number of operating *modes*. Each mode is described by its own set of differential equations. Second, the transition between modes is abrupt, so the system parameters change *discontinuously*. The state variables of the system (i.e. the physical quantities like inductor currents, capacitor voltages, velocities, positions etc.) are the same for all modes, and they are *continuous* in time.

It is assumed that the system is in *cyclic operation*, which means that there is a fixed, repetitive sequence of modes called a *cycle*. By definition, mode i is active in the interval given by $t \in [t_{i-1}[k], t_i[k])$, in which k denotes the k-th cycle. The time instants $t_i[k]$ are *event times* on which a change of modes takes place. This changing of modes over different intervals is illustrated in figure 1. Corresponding modes in different cycles do not have to have the same duration, nor do consecutive cycles have to have the same length. Further, all modes are described by the same type of linear, time-invariant model (1). The parameters of these models, however, vary for each mode. In (1) the matrices A_i ($n \times n$) and B_i ($n \times l$) represent resp. the system matrix and the input matrix for mode i. The state vector $x(t)$ ($n \times 1$) is the same for all modes.

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{1}$$

From this model the behaviour of the state $x(t)$ can be calculated. First the evolution of the state over one mode interval is obtained. Starting from $t = t_{i-1}[k]$, the state vector $x_i[k] = x(t_i[k])$ is found, using some short-hand notations, from

$$x_i[k] = \Phi_i[k] x_{i-1}[k] + \Gamma_i[k] \quad \text{with} \tag{2}$$

$$\Phi_i[k] = e^{A_i(t_i[k]-t_{i-1}[k])}$$

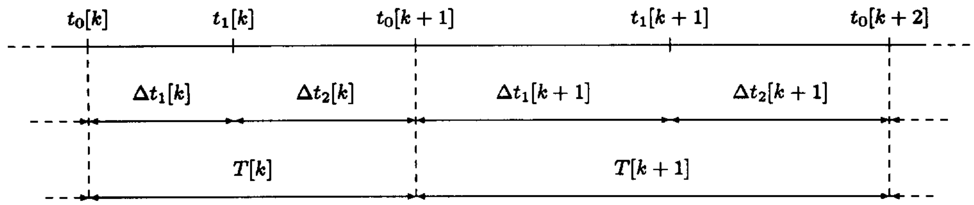

Figure 1. Definition time intervals

$$\Gamma_i[k] = \int_{t_{i-1}[k]}^{t_i[k]} e^{A_i(t_i-\tau)} B_i u(\tau)\, d\tau$$

Calculating the state $x(t)$ over more than one interval is done by applying (2) recursively. For systems fed by DC-power supplies (2) can further be simplified by application of the augmented state vector method introduced in [1], especially if one (or more) of the system matrices A_i is singular. State vector augmentation is done by reformulating the system (1) into

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ 1 \end{pmatrix} = \left(\begin{array}{c|c} A_i & B_i u \\ \hline 0 & 0 \end{array}\right)\begin{pmatrix} x(t) \\ 1 \end{pmatrix} \qquad (3)$$

$$\hat{x}(t) = \hat{A}_i \hat{x}(t) \qquad (4)$$

With state vector augmentation, the evolution of the state vector is also calculated from a previous value of the state vector. The resulting expressions are simpler, however, because the augmented system is homogeneous. For each mode i (2) can be rewritten as

$$\hat{x}_i[k] = e^{\hat{A}_i \Delta t_i[k]} \hat{x}_{i-1}[k] \;=\; \hat{\Phi}_i[k]\hat{x}(t_{i-1}) \qquad (5)$$

Appying (5) recursively gives an expression for the state evolution over a complete cycle.

$$\begin{aligned}\hat{x}_0[k+1] &= \hat{\Phi}_m[k]\cdots\hat{\Phi}_1[k]\hat{x}_0[k] \\ &= \hat{\Phi}_{tot}[k]\hat{x}_0[k] \qquad (6)\end{aligned}$$

Model (6) is a sampled-data model describing the behaviour of the state vector at the beginning of a cycle. This model will further be used to investigate the dynamic properties of the switching system. For notational simplicity $\hat{x}$ and $\hat{\Phi}$ will from this point on be written as x resp. Φ. Augmentation of the state vector is therefore implicitly assumed for all variables.

Transitions

In the previous section the switching instants $t_i[k]$ were assumed to be known. In general, the transitions between modes are governed by a set of transition conditions that explicitly or implicitly determine when a transition takes place. A transition condition can be written as a constraint of the form $g_i[k] = 0$, in which i denotes the transition from mode i to the next mode in cycle k. Each constraint specifies a relation between the transition moment t_i and some triggering signal, which may include control actions, a timing mechanism, a function of the state vector (e.g. a diode current becoming zero) etc. For example, for Pulse Width Modulation (PWM) the duration of the first mode of cycle k, $\Delta t_1[k]$, can be dictated by some external control signal $v[k]$. This results in a constraint $g_1[k] = \Delta t_1[k] - v[k] = 0$. In principle, any function could serve as a transition condition, which would make the analysis of the system difficult. Therefore, the transition conditions are restricted to be linear functions of the state vector x_i, the time intervals Δt_i, an external input term v and a digital linear controller x_c. Grouping the time intervals Δt_i into a vector $d[k] = (\ \Delta t_1[k]\ \ \ldots\ \ \Delta t_m[k]\ )^T$, the transition conditions can be written in a general matrix/vector notation.

$$\begin{aligned}0 = {}& G_x[k]x_0[k] + G_c x_c[k] + G_d d[k] + \\ &+ G_v v[k] + G_0 \qquad (7)\end{aligned}$$

The controller state $x_c[k]$ in (7) is governed by the following difference equation

$$x_c[k+1] = \Phi_c x_c[k] + \Gamma_{cx}[k]x_0[k] + \Gamma_{cd} d[k] \quad (8)$$

In (7) and (8) the parameters $G_x[k]$, G_c, G_d, G_v, G_0, Φ_c, $\Gamma_{cx}[k]$ and Γ_{cd} are matrices of appropriate dimensions. The general formulation of (7)-(8) allows the transition conditions to include almost any linear control scheme. Also, state-dependent mode transitions (e.g. a diode that stops conducting when the current drops to zero) are easily modelled this way.

CYCLIC STATE

In this section the behaviour of the system in cyclic mode will be analysed. Due to the constant changes in system parameters A_i and B_i, the state vector of the system will not converge to a steady-state, but to a *cyclic state*. A system is said to be in cyclic state if the behaviour of the system is periodic, i.e. both the state vector and the mode transitions are periodic. The cyclic state is found by stating that the state vector at the end of a cycle must equal the state at the beginning of the cycle, thus

$$x_0[k+1] = x_0[k] \; = \; x_0 \tag{9}$$

Solving this equation yields the periodic state vector x_0 at the beginning of the cyclic period. Using (6), the cyclic state condition becomes

$$(\Phi_{tot}(d) - I)x_0 = 0 \tag{10}$$

in which I is the identity matrix. This condition can easily be solved if the time intervals d are known (see e.g. [1]). As was shown in the previous section, this is not the case. The time intervals are determined by the transition conditions (7) and (8). In cyclic state, these conditions must still hold, so using the cyclic state values of all variables, the cyclic state transition conditions become

$$0 = G_x(d)x_0 + G_c x_c + G_d d + G_v v + G_0 \tag{11a}$$

$$0 = \Gamma_{cx}(d)x_0 + (\Phi_c - I)x_c + \Gamma_{cd} d \tag{11b}$$

Combining (11) with (10) gives a set of equations from which the cyclic state values of the state x_0, the controller state x_c and time intervals d can be solved given the input v. Notice that the equations are nonlinear in d, but not in x_0 and x_c. This makes it possible to eliminate x_0 and x_c from the equations, yielding a reduced order problem of the form $F(d) = 0$, which can subsequently be solved numerically, e.g. via Newton-Raphson. Substitution of the resulting d in (10)-(11) gives a linear set of equations from which x_0 and x_c can be determined directly.

Using the cyclic state values of x_0 and d and (2)-(5), all values of the cyclic state $x_{cs}(t)$ can be calculated analytically for any t. As described in [1], [2] this cyclic state can be used to calculate average values, harmonics etc. of the state variables.

SMALL-SIGNAL MODEL

In the previous section the behaviour of the system in cyclic state was obtained. The cyclic state is the equivalent of the steady-state in normal (non-switching) systems, so it describes the *static* properties of the system. For stability analysis and control purposes it is also necessary to know the dynamic properties of the system. A dynamical model of the switching system was given by the equations (6)-(8) and describes the full (large-signal) behaviour of the switching in cyclic operation. The problem with this model is that it is nonlinear, which makes it difficult to analyse. Therefore, a description of the system's *local* behaviour around an operating point will be investigated. In the vicinity of an operating point the properties of the system are almost constant. This means that for small deviations of the operating point the nonlinear system model may be approximated by a linearized model (see e.g. [7]), the small-signal model. There are several methods to arrive at a small-signal model. The two most commonly known methods are state-space averaging [3], and exact linearization (e.g. [4], [5], [6]). Here, the latter will be used. For this, the notion of operating point has to be defined first. An operating point of a system is the set of inputs, outputs and states for which the system can be in steady-state. For a switching system the term steady-state must be interpreted as cyclic state. Characteristic for a switching system in cyclic state are the state x_0 at the beginning of a cycle, the controller state x_c, the duration of the mode intervals d and the input v. The operating point is denoted by $P = \{x_0, x_c, d, v\}$.

The exact large-signal model for the behaviour of state vector $x_0[k]$ is given by (6). Linearization of (6) is obtained by taking the first-order Taylor approximation, yielding

$$\Delta x_0[k+1] = \Phi_{tot}(d)\Delta x_0[k] + \nabla_d(\Phi_{tot}x_0)\Delta d[k] \tag{12}$$

in which $\Delta x_0[k] = x_0[k] - x_0$, $\Delta d[k] = d[k] - d$ and ∇_d denotes the gradient operator with respect to d in the operating point P. Each of the columns of the matrix $\nabla_d(\Phi_{tot}x_0) = (\; \gamma_1 \;\; \dots \;\; \gamma_m \;)^T$, with dimensions $\dim n \times m$, is calculated from

$$\gamma_i = \frac{\partial}{\partial d_i}(\Phi_{tot}x_0) \; = \; \Phi_{m,i+1}A_i\Phi_{i,1} \cdot x_0 \tag{13}$$

In (12), the time intervals $d[k]$ were considered independent variables of the linearized system. They are, however, determined by the transition constraints given by (7) and (8). In cyclic state, the operating point variables $\{x_0, x_c, v, d\}$ satisfy the transition constraints. For deviations from the operating point the transition conditions still have to be fulfilled, so any perturbations in the states x_0, x_c and the input v will result in deviations of d from its operating point. To calculate these deviations, the transition constraints are also linearized in the operating point P. Linearization of the transition constraint (7), the evolution of the controller state, yields

$$\begin{aligned}\Delta x_c[k+1] = \Gamma_{cx}(d)\Delta x_0[k] + \Phi_c \Delta x_c[k] + \\ + (\Gamma_{cd} + \nabla_d(\Gamma_{cx}x_0))\Delta d[k] + \Gamma_{cv}\Delta v[k]\end{aligned} \tag{14}$$

in which $\nabla_d(\Gamma_{cx}x_0)$, $\dim n_c \times m$, is obtained analogously to (13). Next, the transition constraint (8), relating the current time intervals $d[k]$ to the states $x_0[k]$ and $x_c[k]$ and the input $v[k]$ is linearized.

$$0 = G_x(d)\Delta x_0[k] + G_c \Delta x_c[k] + \tag{15}$$

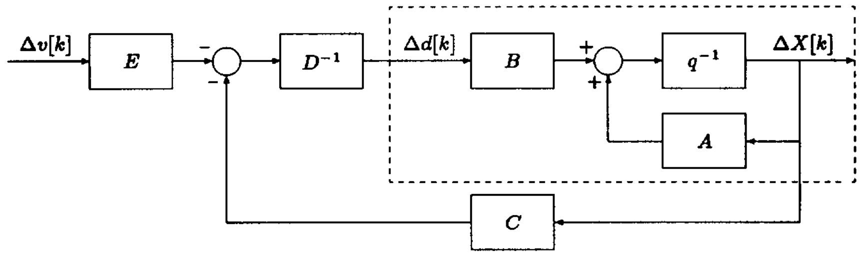

Figure 2. Block diagram of the complete linearized system

$$+(G_d + \nabla_d(G_x x_0))\Delta d[k] + G_v \Delta v[k]$$

Combining the linearized model equations (12), (14) and (15) into one matrix/vector representation gives

$$\begin{pmatrix} \Delta X[k+1] \\ 0 \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \begin{pmatrix} \Delta X[k] \\ \Delta d[k] \end{pmatrix} + \qquad (16)$$
$$+ \begin{pmatrix} 0 \\ E \end{pmatrix} \Delta v[k] \qquad \text{with}$$

$$\Delta X[k] = \begin{pmatrix} \Delta x_0[k] \\ \Delta x_c[k] \end{pmatrix}$$
$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} \Phi_{tot}(d) & 0 & \nabla_d(\Phi_{tot} x_0) \\ \Gamma_{cx}(d) & \Phi_c & \Gamma_{cd} + \nabla_d(\Gamma_{cx} x_0) \\ \hline G_x(d) & G_c & G_d + \nabla_d(G_x x_0) \end{array}\right)$$
$$E = G_v$$

Equation (16) is the relation between all small-signal variables. Eliminating $\Delta d[k]$ from the bottom rows of (16) gives the linearized dynamical model for the combined state vector $\Delta X[k]$. A block diagram of the linearized model is given in figure 2. In the diagram q^{-1} denotes the unit delay operator.

$$\Delta X[k+1] = (A - BD^{-1}C)\Delta X[k] + \qquad (17)$$
$$- (BD^{-1}E)\Delta v[k]$$

Notice that given the cyclic state x_0 all terms of the linearized model can be calculated directly.

EXAMPLE

As an example that the theory can be applied not only to switching electrical networks but also to a more general class of motion systems, a DC-motor drive is chosen. The drive consists of an electronic driver circuit, a filter and a DC-motor. The output of the control system is a PWM signal ($T_{PWM} = 85\mu s$), which is switched between maximum voltage (40 V) and zero with duty ratio α. The current through the motor is restricted to be positive by hardware. The model of the drive system is given by

$$\dot{\omega}(t) = (-f \cdot \omega(t) + c_\varphi \cdot i_L(t))/J$$
$$\dot{i_L}(t) = (u(t) - c_\varphi \cdot \omega(t) - R_m \cdot i_L(t))/L_m$$

with $\omega(t)$ the angular velocity, i_L the motor current and

c_φ	0.04 Nm/A	motor parameter
L_m	0.7 mH	motor inductance
R_m	5.0 Ω	motor resistance
J	$2.0\ 10^{-6}$ kgm²	inertia
f	$3.0\ 10^{-6}$ Nms/rad	friction coefficient

Since the motor current i_L can become zero, there are three possible modes of operation: voltage on, voltage off and voltage off/current zero. This leads to three sets of A and B-matrices and three transition constraints. The transition constraints between the modes are:

$$\begin{array}{ll} g_{1\to 2}[k]: & \Delta t_1[k] = \alpha[k] \cdot T \\ g_{2\to 3}[k]: & i_L(t_2[k]) = 0 \\ g_{3\to 1}[k]: & \Delta t_1[k] + \Delta t_2[k] + \Delta t_3[k] = T \end{array}$$

Rewriting these constraints in the form (7), using an (augmented) state $x(t) = \begin{pmatrix} \omega(t) & i_L(t) & 1 \end{pmatrix}^T$, yields

$$G_x[k] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad G_v = \begin{pmatrix} -T \\ 0 \\ 0 \end{pmatrix}$$
$$G_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad G_0 = \begin{pmatrix} 0 \\ 0 \\ -T \end{pmatrix}$$

The other transition parameters in (7)-(8) are zero.

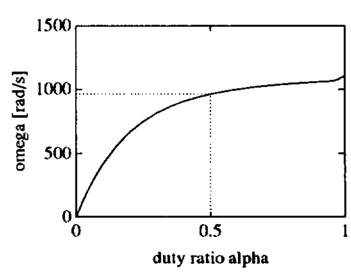

Figure 3. Angular velocity ω versus duty ratio α in cyclic state

Solving the cyclic state conditions for this system yields the relation between the duty ratio α and the cyclic state. Figure 3 shows the nonlinear relation between duty ratio α and angular velocity ω. The gain of the linearized system (=the slope of the curve) varies by a factor of more than 20! This nonlinearity is mainly due to the fact that the motor current i_L becomes zero at some point of the cycle, which strongly depends on the duty ratio α. Figure 4 shows the behaviour of the motor current in cyclic state.

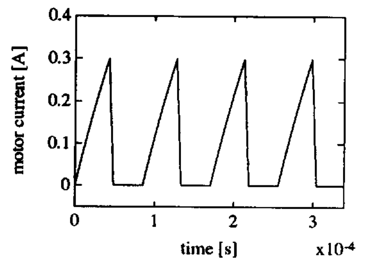

Figure 4. Motor current for $\alpha = 0.5$

Calculating (17) yields a sampled data model of the system. With this model a comparison was made between the exact (nonlinear) solution of the state vector (via simulation) and the linearized model. The operating point was chosen as $\alpha = 0.5$. In the simulation the duty ratio was varied between 0.45 and 0.55. Figure 5 shows that the linearized model (solid line) is a good approximation of the exact solution (dashed line), which is also confirmed by measurements.

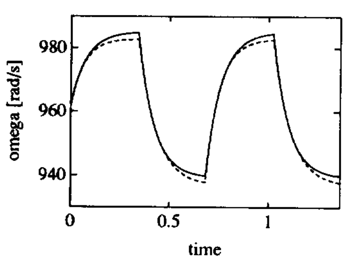

Figure 5. Comparison exact and linearized model

CONCLUSIONS

Modelling of the system systems with an augmented state vector yields concise models which can easily be manipulated. The transition conditions between the modes of the switching system are formulated in a structured way, allowing most practical (linear) control schemes to be included directly. From this system formulation the cyclic state of the system is obtained. Also, linear dynamic models of the small-signal behaviour are easily calculated using this formulation. Simulations show that the small-signal model gives a good description of the system behaviour.

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