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Voltage and Current Laws 3.1

Michael D. Ciletti

Analysis of linear circuits rests on two fundamental physical laws that describe how the voltages and currents in a circuit must behave. This behavior results from whatever voltage sources, current sources, and energy storage elements are connected to the circuit. A voltage source imposes a constraint on the evolution of the voltage between a pair of nodes; a current source imposes a constraint on the evolution of the current in a branch of the circuit. The energy storage elements (capacitors and inductors) impose initial conditions on currents and voltages in the circuit; they also establish a dynamic relationship between the voltage and the current at their terminals.

Regardless of how a linear circuit is stimulated, every node voltage and every branch current, at every instant of time, must be consistent with Kirchhoff's voltage and current laws. These two laws govern even the most complex linear circuits. (They also apply to a broad category of nonlinear circuits that are modeled by point models of voltage and current.)

A circuit can be considered to have a topological (or graph) view, consisting of a labeled set of nodes and a labeled set of edges. Each edge is associated with a pair of nodes. A node is drawn as a dot and represents a



FIGURE 3.1 Graph representation of a linear circuit.

connection between two or more physical components; an edge is drawn as a *line* and represents a path, or branch, for current flow through a component (see Fig. 3.1).

The edges, or branches, of the graph are assigned current labels, i_1, i_2, \ldots, i_m . Each current has a designated direction, usually denoted by an *arrow* symbol. If the arrow is drawn toward a node, the associated current is said to be *entering* the node; if the arrow is drawn away from the node, the current is said to be *leaving* the node. The current i_1 is entering node b in Fig. 3.1; the current i_5 is leaving node e.

Given a branch, the pair of nodes to which the branch is attached defines the convention for measuring voltages in the circuit. Given the ordered pair of nodes (a, b), a voltage measurement is formed as follows:

$$v_{ab} = v_a - v_b$$

where v_a and v_b are the absolute electrical potentials (voltages) at the respective nodes, taken relative to some reference node. Typically, one node of the circuit is labeled as *ground*, or reference node; the remaining nodes are assigned voltage labels. The measured quantity, v_{ab} , is called the *voltage drop* from node *a* to node *b*. We note that

$$v_{ab} = -v_{ba}$$

and that

$$v_{ba} = v_b - v_a$$

is called the *voltage rise* from *a* to *b*. Each node voltage implicitly defines the voltage drop between the respective node and the ground node.

The pair of nodes to which an edge is attached may be written as (a, b) or (b, a). Given an ordered pair of nodes (a, b), a *path from a to b* is a directed sequence of edges in which the first edge in the sequence contains node label *a*, the last edge in the sequence contains node label *b*, and the node indices of any two adjacent members of the sequence have at least one node label in common. In Fig. 3.1, the edge sequence $\{e_1, e_2, e_4\}$ is not a path, because e_2 and e_4 do not share a common node label. The sequence $\{e_1, e_2\}$ is a path from node *a* to node *c*.

A path is said to be *closed* if the first node index of its first edge is identical to the second node index of its last edge. The following edge sequence forms a closed path in the graph given in Fig. 3.1: $\{e_1, e_2, e_3, e_4, e_6, e_7\}$. Note that the edge sequences $\{e_8\}$ and $\{e_1, e_1\}$ are closed paths.

Kirchhoff's Current Law

Kirchhoff's current law (KCL) imposes constraints on the currents in the branches that are attached to each node of a circuit. In simplest terms, KCL states that the sum of the currents that are entering a given node

must equal the sum of the currents that are leaving the node. Thus, the set of currents in branches attached to a given node can be partitioned into two groups whose orientation is away from (into) the node. The two groups must contain the same net current. Applying KCL at node b in Fig. 3.1 gives

$$i_1(t) + i_3(t) = i_2(t)$$

A connection of water pipes that has no leaks is a physical analogy of this situation. The net rate at which water is flowing into a joint of two or more pipes must equal the net rate at which water is flowing away from the joint. The joint itself has the property that it only connects the pipes and thereby imposes a structure on the flow of water, but it cannot store water. This is true regardless of when the flow is measured. Likewise, the nodes of a circuit are modeled as though they cannot store charge, (Physical circuits are sometimes modeled for the purpose of simulation as though they store charge, but these nodes implicitly have a capacitor that provides the physical mechanism for *storing* the charge. Thus, KCL is ultimately satisfied.)

KCL can be stated alternatively as: "the algebraic sum of the branch currents entering (or leaving) any node of a circuit at any instant of time must be zero." In this form, the label of any current whose orientation is away from the node is preceded by a minus sign. The currents *entering* node *b* in Fig. 3.1 must satisfy

$$i_1(t) - i_2(t) + i_3(t) = 0$$

In general, the currents entering or leaving each node *m* of a circuit must satisfy

$$\sum i_{km}(t) = 0$$

where $i_{km}(t)$ is understood to be the current in branch k attached to node m. The currents used in this expression are understood to be the currents that would be measured in the branches attached to the node, and their values include a magnitude and an algebraic sign. If the measurement convention is oriented for the case where currents are entering the node, then the actual current in a branch has a positive or negative sign, depending on whether the current is truly flowing toward the node in question.

Once KCL has been written for the nodes of a circuit, the equations can be rewritten by substituting into the equations the voltage-current relationships of the individual components. If a circuit is resistive, the resulting equations will be algebraic. If capacitors or inductors are included in the circuit, the substitution will produce a differential equation. For example, writing KCL at the node for v_3 in Fig. 3.2 produces

$$i_2 + i_1 - i_3 = 0$$

and

$$C_1 \frac{dv_1}{dt} + \frac{v_4 - v_3}{R_2} - C_2 \frac{dv_2}{dt} = 0$$



FIGURE 3.2 Example of a circuit containing energy storage elements.

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KCL for the node between C_2 and R_1 can be written to eliminate variables and lead to a solution describing the capacitor voltages. The capacitor voltages, together with the applied voltage source, determine the remaining voltages and currents in the circuit. Nodal analysis (see Section 3.2) treats the systematic modeling and analysis of a circuit under the influence of its sources and energy storage elements.

Kirchhoff's Current Law in the Complex Domain

Kirchhoff's current law is ordinarily stated in terms of the real (time-domain) currents flowing in a circuit, because it actually describes physical quantities, at least in a macroscopic, statistical sense. It also applied, however, to a variety of purely mathematical models that are commonly used to analyze circuits in the so-called complex domain.

For example, if a linear circuit is in the sinusoidal steady state, all of the currents and voltages in the circuit are sinusoidal. Thus, each voltage has the form

$$v(t) = A \sin(\omega t + \phi)$$

and each current has the form

$$i(t) = B \sin(\omega t + \theta)$$

where the positive coefficients A and B are called the magnitudes of the signals, and ϕ and θ are the phase angles of the signals. These mathematical models describe the physical behavior of electrical quantities, and instrumentation, such as an oscilloscope, can display the actual waveforms represented by the mathematical model. Although methods exist for manipulating the models of circuits to obtain the magnitude and phase coefficients that uniquely determine the waveform of each voltage and current, the manipulations are cumbersome and not easily extended to address other issues in circuit analysis.

Steinmetz [Smith and Dorf, 1992] found a way to exploit complex algebra to create an elegant framework for representing signals and analyzing circuits when they are in the steady state. In this approach, a model is developed in which each physical sign is replaced by a "complex" mathematical signal. This complex signal in polar, or exponential, form is represented as

$$v_c(t) = Ae^{(j\omega t + \phi)}$$

The algebra of complex exponential signals allows us to write this as

$$v_c(t) = A e^{j\phi} e^{j\omega t}$$

and Euler's identity gives the equivalent rectangular form:

$$v_{c}(t) = A[\cos(\omega t + \phi) + j\sin(\omega t + \phi)]$$

So we see that a physical signal is either the real (cosine) or the imaginary (sine) component of an abstract, complex mathematical signal. The additional mathematics required for treatment of complex numbers allows us to associate a phasor, or complex amplitude, with a sinusoidal signal. The time-invariant phasor associated with v(t) is the quantity

$$\mathbf{V}_{c} = Ae^{j\phi}$$

Notice that the phasor \mathbf{v}_c is an algebraic constant and that in incorporates the parameters A and ϕ of the corresponding time-domain sinusoidal signal.

Phasors can be thought of as being vectors in a two-dimensional plane. If the vector is allowed to rotate about the origin in the counterclockwise direction with frequency ω , the projection of its tip onto the horizontal

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(real) axis defines the time-domain signal corresponding to the real part of $v_c(t)$, i.e., $A \cos[\omega t + \phi]$, and its projection onto the vertical (imaginary) axis defines the time-domain signal corresponding to the imaginary part of $v_c(t)$, i.e., $A \sin[\omega t + \phi]$.

The composite signal $v_c(t)$ is a mathematical entity; it cannot be seen with an oscilloscope. Its value lies in the fact that when a circuit is in the steady state, its voltages and currents are uniquely determined by their corresponding phasors, and these in turn satisfy Kirchhoff's voltage and current laws! Thus, we are able to write

$$\sum I_{km} = 0$$

where I_{km} is the phasor of $i_{km}(t)$, the sinusoidal current in branch k attached to node m. An equation of this form can be written at each node of the circuit. For example, at node b in Fig. 3.1 KCL would have the form

$$I_1 - I_2 + I_3 = 0$$

Consequently, a set of linear, algebraic equations describe the phasors of the currents and voltages in a circuit in the sinusoidal steady state, i.e., the notion of time is suppressed (see Section 3.2). The solution of the set of equations yields the phasor of each voltage and current in the circuit, from which the actual time-domain expressions can be extracted.

It can also be shown that KCL can be extended to apply to the Fourier transforms and the Laplace transforms of the currents in a circuit. Thus, a single relationship between the currents at the nodes of a circuit applies to all of the known mathematical representations of the currents [Ciletti, 1988].

Kirchhoff's Voltage Law

Kirchhoff's voltage law (KVL) describes a relationship among the voltages measured across the branches in any closed, connected path in a circuit. Each branch in a circuit is connected to two nodes. For the purpose of applying KVL, a path has an orientation in the sense that in "walking" along the path one would enter one of the nodes and exit the other. This establishes a direction for determining the voltage across a branch in the path: the voltage is the difference between the potential of the node entered and the potential of the node at which the path exits. Alternatively, the voltage drop along a branch is the difference of the node voltage at the entered node and the node voltage at the exit node. For example, if a path includes a branch between node "a" and node "b", the voltage drop measured along the path in the direction from node "a" to node "b" is denoted by v_{ab} and is given by $v_{ab} = v_a - v_b$. Given v_{ab} , branch voltage along the path in the direction from node "b" to node "a" is $v_{ba} = v_b - v_a = -v_{ab}$.

Kirchhoff's voltage law, like Kirchhoff's current law, is true at any time. KVL can also be stated in terms of voltage rises instead of voltage drops.

KVL can be expressed mathematically as "the algebraic sum of the voltages drops around any closed path of a circuit at any instant of time is zero." This statement can also be cast as an equation:

$$\sum v_{km}(t) = 0$$

where $v_{km}(t)$ is the instantaneous voltage drop measured across branch k of path m. By convention, the voltage drop is taken in the direction of the edge sequence that forms the path.

The edge sequence $\{e_1, e_2, e_3, e_4, e_6, e_7\}$ forms a closed path in Fig. 3.1. The sum of the voltage drops taken around the path must satisfy KVL:

$$v_{ab}(t) + v_{bc}(t) + v_{cd}(t) + v_{de}(t) + v_{ef}(t) + v_{fa}(t) = 0$$

Since $v_{af}(t) = -v_{fa}(t)$, we can also write

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$$v_{af}(t) = v_{ab}(t) + v_{bc}(t) + v_{cd}(t) + v_{de}(t) + v_{ef}(t)$$

Had we chosen the path corresponding to the edge sequence $\{e_1, e_5, e_6, e_7\}$ for the path, we would have obtained

$$v_{af}(t) = v_{ab}(t) + v_{be}(t) + v_{ef}(t)$$

This demonstrates how KCL can be used to determine the voltage between a pair of nodes. It also reveals the fact that the voltage between a pair of nodes is independent of the path between the nodes on which the voltages are measured.

Kirchhoff's Voltage Law in the Complex Domain

Kirchhoff's voltage law also applies to the phasors of the voltages in a circuit in steady state and to the Fourier transforms and Laplace transforms of the voltages in a circuit.

Importance of KVL and KCL

Kirchhoff's current law is used extensively in nodal analysis because it is amenable to computer-based implementation and supports a systematic approach to circuit analysis. Nodal analysis leads to a set of algebraic equations in which the variables are the voltages at the nodes of the circuit. This formulation is popular in CAD programs because the variables correspond directly to physical quantities that can be measured easily.

Kirchhoff's voltage law can be used to completely analyze a circuit, but it is seldom used in large-scale circuit simulation programs. The basic reason is that the currents that correspond to a loop of a circuit do not necessarily correspond to the currents in the individual branches of the circuit. Nonetheless, KVL is frequently used to troubleshoot a circuit by measuring voltage drops across selected components.

Defining Terms

Branch: A symbol representing a path for current through a component in an electrical circuit.

Branch current: The current in a branch of a circuit.

Branch voltage: The voltage across a branch of a circuit.

Independent source: A voltage (current) source whose voltage (current) does not depend on any other voltage or current in the circuit.

Node: A symbol representing a physical connection between two electrical components in a circuit. **Node voltage:** The voltage between a node and a reference node (usually ground).

Related Topic

3.6 Graph Theory

References

M.D. Ciletti, *Introduction to Circuit Analysis and Design*, New York: Holt, Rinehart and Winston, 1988. R.H. Smith and R.C. Dorf, *Circuits, Devices and Systems*, New York: Wiley, 1992.

Further Information

Kirchhoff's laws form the foundation of modern computer software for analyzing electrical circuits. The interested reader might consider the use of determining the minimum number of algebraic equations that fully characterizes the circuit. It is determined by KCL, KVL, or some mixture of the two?

J. David Irwin

In this section Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL) will be used to determine currents and voltages throughout a network. For simplicity, we will first illustrate the basic principles of both node analysis and mesh analysis using only dc circuits. Once the fundamental concepts have been explained and illustrated, we will demonstrate the generality of both analysis techniques through an ac circuit example.

Node Analysis

In a node analysis, the node voltages are the variables in a circuit, and KCL is the vehicle used to determine them. One node in the network is selected as a reference node, and then all other node voltages are defined with respect to that particular node. This reference node is typically referred to as *ground* using the symbol (\pm) , indicating that it is at ground-zero potential.

Consider the network shown in Fig. 3.3. The network has three nodes, and the nodes at the bottom of the circuit has been selected as the reference node. Therefore the two remaining nodes, labeled V_1 and V_2 , are measured with respect to this reference node.



FIGURE 3.3 A three-node network.

Suppose that the node voltages V_1 and V_2 have somehow been determined, i.e., $V_1 = 4$ V and $v_2 = -4$ V. Once these node voltages are known, Ohm's law can be used to find all branch currents. For example,

$$I_{1} = \frac{V_{1} - 0}{2} = 2 A$$

$$I_{2} = \frac{V_{1} - V_{2}}{2} = \frac{4 - (-4)}{2} = 4 A$$

$$I_{3} = \frac{V_{2} - 0}{1} = \frac{-4}{1} = -4 A$$

Note that KCL is satisfied at every node, i.e.,

$$I_1 - 6 + I_2 = 0$$
$$-I_2 + 8 + I_3 = 0$$
$$-I_1 + 6 - 8 - I_3 = 0$$

Therefore, as a general rule, if the node voltages are known, all branch currents in the network can be immediately determined.

In order to determine the node voltages in a network, we apply KCL to every node in the network except the reference node. Therefore, given an *N*-node circuit, we employ N-1 linearly independent simultaneous equations to determine the N-1 unknown node voltages. Graph theory, which is covered in Section 3.6, can be used to prove that exactly N-1 linearly independent KCL equations are required to find the N-1 unknown node voltages in a network.

Let us now demonstrate the use of KCL in determining the node voltages in a network. For the network shown in Fig. 3.4, the bottom



FIGURE 3.4 A four-node network.

node is selected as the reference and the three remaining nodes, labeled V_1 , V_2 , and V_3 , are measured with respect to that node. All unknown branch currents are also labeled. The KCL equations for the three nonreference nodes are

$$I_1 + 4 + I_2 = 0$$

-4 + I₃ + I₄ = 0
-I₁ - I₄ - 2 = 0

Using Ohm's law these equations can be expressed as

$$\frac{V_1 - V_3}{2} + 4 + \frac{V_1}{2} = 0$$
$$-4 + \frac{V_2}{1} + \frac{V_2 - V_3}{1} = 0$$

$$-\frac{(V_1 - V_3)}{2} - \frac{(V_2 - V_3)}{1} - 2 = 0$$

Solving these equations, using any convenient method, yields $V_1 = -8/3$ V, $V_2 = 10/3$ V, and $V_3 = 8/3$ V. Applying Ohm's law we find that the branch currents are $I_1 = -16/6$ A, $I_2 = -8/6$ A, $I_3 = 20/6$ A, and $I_4 = 4/6$ A. A quick check indicates that KCL is satisfied at every node.





FIGURE 3.5 A four-node network containing voltage sources.

node and nodes V_1 and V_3 , these two node voltages are known, i.e., $V_1 = 12$ V and $V_3 = -4$ V. Therefore, we have only one unknown node voltage, V_2 . The equations for this network are then

$$V_1 = 12$$
$$V_3 = -4$$

and

 $-I_1 + I_2 + I_3 = 0$

The KCL equation for node V_2 written using Ohm's law is

$$-\frac{(12-V_2)}{1} + \frac{V_2}{2} + \frac{V_2 - (-4)}{2} = 0$$

Solving this equation yields $V_2 = 5$ V, $I_1 = 7$ A, $I_2 = 5/2$ A, and $I_3 = 9/2$ A. Therefore, KCL is satisfied at every node.

Thus, the presence of a voltage source in the network actually simplifies a node analysis. In an attempt to generalize this idea, consider the network in Fig. 3.6. Note that in this case $V_1 = 12$ V and the difference between node voltages V_3 and V_2 is constrained to be 6 V. Hence, two of the three equations needed to solve for the node voltages in the network are

$$V_1 = 12$$
$$V_3 - V_2 = 6$$



FIGURE 3.6 A four-node network used to illustrate a supernode.

To obtain the third required equation, we form what is called a

supernode, indicated by the dotted enclosure in the network. Just as KCL must be satisfied at any node in the network, it must be satisfied at the supernode as well. Therefore, summing all the currents leaving the supernode yields the equation

$$\frac{V_2 - V_1}{1} + \frac{V_2}{2} + \frac{V_3 - V_1}{1} + \frac{V_3}{2} = 0$$

The three equations yield the node voltages $V_1 = 12$ V, $V_2 = 5$ V, and $V_3 = 11$ V, and therefore $I_1 = 1$ A, $I_2 = 7$ A, $I_3 = 5/2$ A, and $I_4 = 11/2$ A.

Mesh Analysis

In a mesh analysis the mesh currents in the network are the variables and KVL is the mechanism used to determine them. Once all the mesh currents have been determined, Ohm's law will yield the voltages anywhere in e circuit. If the network contains N independent meshes, then graph theory can be used to prove that N independent linear simultaneous equations will be required to determine the N mesh currents.

The network shown in Fig. 3.7 has two independent meshes. They are labeled I_1 and I_2 , as shown. If the mesh currents are known to be $I_1 = 7$ A and $I_2 = 5/2$ A, then all voltages in the network can be calculated. For example, the voltage V_{12} i.e., the voltage across the 1- Ω



FIGURE 3.7 A network containing two independent meshes.

resistor, is $V_1 = -I_1R = -(7)(1) = -7$ V. Likewise $V = (I_1 - I_2)R = (7 - 5/2)(2) = 9$ V. Furthermore, we can check our analysis by showing that KVL is satisfied around every mesh. Starting at the lower left-hand corner and applying KVL to the left-hand mesh we obtain

$$-(7)(1) + 16 - (7 - 5/2)(2) = 0$$

where we have assumed that increases in energy level are positive and decreases in energy level are negative.

Consider now the network in Fig. 3.8. Once again, if we assume that an increase in energy level is positive and a decrease in energy level is negative, the three KVL equations for the three meshes defined are

$$-I_1(1) - 6 - (I_1 - I_2)(1) = 0$$

+12 - (I_2 - I_1)(1) - (I_2 - I_3)(2) = 0
-(I_3 - I_2)(2) + 6 - I_3(2) = 0



FIGURE 3.8 A three-mesh network.

These equations can be written as

$$2I_1 - I_2 = -6$$
$$-I_{12} + 3I_2 - 2I_3 = 12$$
$$- 2I_2 + 4I_3 = 6$$

Solving these equations using any convenient method yields $I_1 = 1$ A, $I_2 = 8$ A, and $I_3 = 11/2$ A. Any voltage in the network can now be easily calculated, e.g., $V_2 = (I_2 - I_3)(2) = 5$ V and $V_3 = I_3(2) = 11$ V.

Just as in the node analysis discussion, we now expand our capabilities by considering circuits which contain current sources. In this case, we will show that for mesh analysis, the presence of current sources makes the solution easier.

The network in Fig. 3.9 has four meshes which are labeled I_1 , I_2 , I_3 , and I_4 . However, since two of these currents, i.e., I_3 and I_4 , pass directly through a current source, two of the four linearly independent equations required to solve the network are





FIGURE 3.9 A four-mesh network containing current sources.

The two remaining KVL equations for the meshes defined by I_1 and I_2 are

$$+6 - (I_1 - I_2)(1) - (I_1 - I_3)(2) = 0$$
$$-(I_2 - I_1)(1) - I_2(2) - (I_2 - I_4)(1) = 0$$

Solving these equations for I_1 and I_2 yields $I_1 = 54/11$ A and $I_2 = 8/11$ A. A quick check will show that KCL is satisfied at every node. Furthermore, we can calculate any node voltage in the network. For example, $V_3 = (I_3 - I_4)(1) = 6$ V and $V_1 = V_3 + (I_1 - I_2)(1) = 112/11$ V.

Summary

Both node analysis and mesh analysis have been presented and discussed. Although the methods have been presented within the framework of dc circuits with only independent sources, the techniques are applicable to ac analysis and circuits containing dependent sources.

To illustrate the applicability of the two techniques to ac circuit analysis, consider the network in Fig. 3.10. All voltages and currents are phasors and the impedance of each passive element is known.

In the node analysis case, the voltage V_4 is known and the voltage between V_2 and V_3 is constrained. Therefore, two of the four required equations are

$$V_4 = 12 / 0^\circ$$

 $_2 + 6 / 0^\circ = V_3$



FIGURE 3.10 A network containing five nodes and four meshes.

KCL for the node labeled V_1 and the supernode containing the nodes labeled V_2 and V_3 is

V

$$\frac{V_1 - V_3}{2} + \frac{V_1 - V_4}{-j1} = 2 / \underline{0}^{\circ}$$
$$\frac{V_2}{1} + 2 / \underline{0}^{\circ} + \frac{V_3 - V_1}{2} + \frac{V_3 - V_4}{j2} = 4 / \underline{0}^{\circ} = 0$$

Solving these equations yields the remaining unknown node voltages.

$$\mathbf{V}_1 = 11.9 - j0.88 = 11.93 / -4.22^\circ \text{V}$$

 $\mathbf{V}_2 = 3.66 - j1.07 = 3.91 / -16.34^\circ \text{V}$
 $\mathbf{V}_3 = 9.66 - j1.07 = 9.72 / -6.34^\circ \text{V}$

In the mesh analysis case, the currents I_1 and I_3 are constrained to be

$$\mathbf{I}_1 = 2 \ \underline{/0^\circ}$$
$$\mathbf{I}_4 - \mathbf{I}_3 = -4 \ \underline{/0^\circ}$$

The two remaining KVL equations are obtained from the mesh defined by mesh current I_2 and the loop which encompasses the meshes defined by mesh currents I_3 and I_4 .

$$-2(I_2 - I_1) - (-j1)I_2 - j2(I_2 - I4) = 0$$

-(1I_3 + 6 /0° - j2(I_4 - I_2) - 12 /0° = 0

Solving these equations yields the remaining unknown mesh currents

$$I_2 = 0.88 / -6.34^{\circ} A$$

 $I_3 = 3.91 / 163.66^{\circ} A$
 $I_4 = 1.13 / 72.35^{\circ} A$

As a quick check we can use these currents to compute the node voltages. For example, if we calculate

$$V_2 = -1(I_3)$$

and

$$V_1 = -j1(I_2) + 12 / 0^\circ$$

we obtain the answers computed earlier.

As a final point, because both node and mesh analysis will yield all currents and voltages in a network, which technique should be used? The answer to this question depends upon the network to be analyzed. If the network contains more voltage sources than current sources, node analysis might be the easier technique. If, however, the network contains more current sources than voltage sources, mesh analysis may be the easiest approach.

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ENGINE-STARTING DEVICE

Charles F. Kettering Patented August 17, 1915 #1,150,523

E arly automobiles were all started with a crank, or arm-strong starters, as they were known. This backbreaking process was difficult for everyone, especially women. And it was dangerous. Backfires often resulted in broken wrists. Worse yet, if accidentally left in gear, the car could advance upon the person cranking. Numerous deaths and injuries were reported.

In 1910, Henry Leland, Cadillac Motors president, commissioned Charles Kettering and his Dayton Engineering Laboratories Company to develop an electric self-starter to replace the crank. Kettering had to overcome two large problems: (1) making a motor small enough to fit in a car yet powerful enough to crank the engine, and (2) finding a battery more powerful than any yet in existence. Electric Storage Battery of Philadelphia supplied an experimental 65-lb battery and Delco unveiled a working prototype electric "self-starter" system installed in a 1912 Cadillac on February 17, 1911. Leland immediately ordered 12,000 units for Cadillac. Within a few years, almost all new cars were equipped with electric starters. (Copyright © 1995, DewRay Products, Inc. Used with permission.)



Defining Terms

- ac: An abbreviation for alternating current.
- dc: An abbreviation for direct current.
- **Kirchhoff's current law (KCL):** This law states that the algebraic sum of the currents either entering or leaving a node must be zero. Alternatively, the law states that the sum of the currents entering a node must be equal to the sum of the currents leaving that node.
- **Kirchhoff's voltage law (KVL):** This law states that the algebraic sum of the voltages around any loop is zero. A loop is any closed path through the circuit in which no node is encountered more than once.
- **Mesh analysis:** A circuit analysis technique in which KVL is used to determine the mesh currents in a network. A mesh is a loop that does not contain any loops within it.

Node analysis: A circuit analysis technique in which KCL is used to determine the node voltages in a network.

- **Ohm's law:** A fundamental law which states that the voltage across a resistance is directly proportional to the current flowing through it.
- **Reference node:** One node in a network that is selected to be a common point, and all other node voltages are measured with respect to that point.
- **Supernode:** A cluster of node, interconnected with voltage sources, such that the voltage between any two nodes in the group is known.

Related Topics

3.1 Voltage and Current Laws • 3.6 Graph Theory

Reference

J.D. Irwin, Basic Engineering Circuit Analysis, 5th ed., Upper Saddle River, N.J.: Prentice-Hall, 1996.

3.3 Network Theorems

Allan D. Kraus

Linearity and Superposition

Linearity

Consider a system (which may consist of a single network element) represented by a block, as shown in Fig. 3.11, and observe that the system has an input designated by e (for excitation) and an output designated by r (for response). The system is considered to be *linear* if it satisfies the *homogeneity* and *superposition* conditions.

The homogeneity condition: If an arbitrary input to the system, *e*, causes a response, *r*, then if *ce* is the input, the output is *cr* where *c* is some arbitrary constant.

The superposition condition: If the input to the system, e_1 , causes a response, r_1 , and if an input to the system, e_2 , causes a response, r_2 , then a response, $r_1 + r_2$, will occur when the input is $e_1 + e_2$.

If neither the homogeneity condition nor the superposition condition is satisfied, the system is said to be *nonlinear*.

The Superposition Theorem

While both the homogeneity and superposition conditions are necessary for linearity, the superposition condition, in itself, provides the basis for the superposition theorem:

If cause and effect are linearly related, the total effect due to several causes acting simultaneously is equal to the sum of the individual effects due to each of the causes acting one at a time.







FIGURE 3.12 (a) A network to be solved by using superposition; (b) the network with the current source nulled; and (c) the network with the voltage source nulled.

Example 3.1. Consider the network driven by a current source at the left and a voltage source at the top, as shown in Fig. 3.12(a). The current phasor indicated by \hat{I} is to be determined. According to the superposition theorem, the current \hat{I} will be the sum of the two current components \hat{I}_V due to the voltage source acting alone as shown in Fig. 3.12(b) and \hat{I}_C due to the current source acting alone shown in Fig. 3.12(c).

$$\hat{I} = \hat{I}_V + \hat{I}_C$$

Figures 3.12(b) and (c) follow from the methods of removing the effects of independent voltage and current sources. Voltage sources are nulled in a network by replacing them with short circuits and current sources are nulled in a network by replacing them with open circuits.

The networks displayed in Figs. 3.12(b) and (c) are simple ladder networks in the phasor domain, and the strategy is to first determine the equivalent impedances presented to the voltage and current sources. In Fig. 3.12(b), the group of three impedances to the right of the voltage source are in series-parallel and possess an impedance of

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$$Z_{P} = \frac{(40 - j40)(j40)}{40 + j40 - j40} = 40 + j40 \,\Omega$$

and the total impedance presented to the voltage source is

$$Z = Z_P + 40 - j40 = 40 + j40 + 40 - j40 = 80 \ \Omega$$

Then \hat{I}_1 , the current leaving the voltage source, is

$$\hat{I}_1 = \frac{240 + j0}{80} = 3 + j0 \,\text{A}$$

and by a current division

$$\hat{I}_V = \left[\frac{j40}{40 - j40 + j40}\right](3 + j0) = j(3 + j0) = 0 + j3A$$

In Fig. 3.12(b), the current source delivers current to the 40- Ω resistor and to an impedance consisting of the capacitor and Z_p . Call this impedance Z_a so that

$$Z_a = -j40 + Z_P = -j40 + 40 + j40 = 40 \ \Omega$$

Then, two current divisions give \hat{I}_C

$$\hat{I}_C = \left[\frac{40}{40+40}\right] \left[\frac{j40}{40-j40+j40}\right] (0-j6) = \frac{j}{2}(0-j6) = 3+j0 \text{ A}$$

The current \hat{I} in the circuit of Fig. 3.12(a) is

$$\hat{I} = \hat{I}_V + \hat{I}_C = 0 + j3 + (3 + j0) = 3 + j3 \text{ A}$$

The Network Theorems of Thévenin and Norton

If interest is to be focused on the voltages and across the currents through a small portion of a network such as network *B* in Fig. 3.13(a), it is convenient to replace network *A*, which is complicated and of little interest, by a simple equivalent. The simple equivalent may contain a single, equivalent, voltage source in series with an equivalent impedance in series as displayed in Fig. 3.13(b). In this case, the equivalent is called a *Thévenin equivalent*. Alternatively, the simple equivalent may consist of an equivalent current source in parallel with an equivalent impedance. This equivalent, shown in Fig. 3.13(c), is called a *Norton equivalent*. Observe that as long as Z_T (subscript *T* for Thévenin) is equal to Z_N (subscript *N* for Norton), the two equivalents may be obtained from one another by a simple source transformation.

Conditions of Application

The Thévenin and Norton network equivalents are only valid at the terminals of network A in Fig. 3.13(a) and they do not extend to its interior. In addition, there are certain restrictions on networks A and B. Network A may contain only linear elements but may contain both independent and dependent sources. Network B, on the other hand, is not restricted to linear elements; it may contain nonlinear or time-varying elements and may



FIGURE 3.13 (a) Two one-port networks; (b) the Thévenin equivalent for network *a*; and (c) the Norton equivalent for network *a*.

also contain both independent and dependent sources. Together, there can be no controlled source coupling or magnetic coupling between networks *A* and *B*.

The Thévenin Theorem

The statement of the Thévenin theorem is based on Fig. 3.13(b):

Insofar as a load which has no magnetic or controlled source coupling to a one-port is concerned, a network containing linear elements and both independent and controlled sources may be replaced by an ideal voltage source of strength, $\hat{V}_{T,}$, and an equivalent impedance Z_T , in series with the source. The value of \hat{V}_T is the open-circuit voltage, \hat{V}_{OO} appearing across the terminals of the network and Z_T is the driving point impedance at the terminals of the network, obtained with all independent sources set equal to zero.

The Norton Theorem

The Norton theorem involves a current source equivalent. The statement of the Norton theorem is based on Fig. 3.13(c):

Insofar as a load which has no magnetic or controlled source coupling to a one-port is concerned, the network containing linear elements and both independent and controlled sources may be replaced by an ideal current source of strength, \hat{I}_N , and an equivalent impedance, Z_N , in parallel with the source. The value of \hat{I}_N is the short-circuit current, \hat{I}_{SC} , which results when the terminals of the network are shorted and Z_N is the driving point impedance at the terminals when all independent sources are set equal to zero.

The Equivalent Impedance, $Z_T = Z_N$

Three methods are available for the determination of Z_T . All of them are applicable at the analyst's discretion. When controlled sources are present, however, the first method cannot be used.

The first method involves the direct calculation of $Z_{eq} = Z_T = Z_N$ by looking into the terminals of the network after all independent sources have been nulled. Independent sources are nulled in a network by replacing all independent voltage sources with a short circuit and all independent current sources with an open circuit.

The second method, which may be used when controlled sources are present in the network, requires the computation of both the Thévenin equivalent voltage (the open-circuit voltage at the terminals of the network) and the Norton equivalent current (the current through the short-circuited terminals of the network). The equivalent impedance is the ratio of these two quantities

$$Z_T = Z_N = Z_{eq} = \frac{\hat{V}_T}{\hat{I}_N} = \frac{\hat{V}_{OC}}{\hat{I}_{SC}}$$

The third method may also be used when controlled sources are present within the network. A test voltage may be placed across the terminals with a resulting current calculated or measured. Alternatively, a test current may be injected into the terminals with a resulting voltage determined. In either case, the equivalent resistance can be obtained from the value of the ratio of the test voltage \hat{V}_o to the resulting current \hat{I}_o

$$Z_T = \frac{\hat{V}_o}{\hat{I}_o}$$

Example 3.2. The current through the capacitor with impedance $-j35 \Omega$ in Fig. 3.14(a) may be found using Thévenin's theorem. The first step is to remove the $-j35 \Omega$ capacitor and consider it as the load. When this is done, the network in Fig. 3.14(b) results.

The Thévenin equivalent voltage is the voltage across the 40– Ω resistor. The current through the 40- Ω resistor was found in Example 3.1 to be $I = 3 + j3 \Omega$. Thus,

$$\hat{V}_T = 40(3 + j3) = 120 + j120 \text{ V}$$

The Thévenin equivalent impedance may be found by looking into the terminals of the network in Fig. 3.14(c). Observe that both sources in Fig. 3.14(a) have been nulled and that, for ease of computation, impedances Z_a and Z_b have been placed on Fig. 3.14(c). Here,

$$Z_{a} = \frac{(40 - j40)(j40)}{40 + j40 - j40} = 40 + j40 \Omega$$
$$Z_{b} = \frac{(40)(40)}{40 + 40} = 20 \Omega$$

and

$$Z_T = Z_b + j15 = 20 + j15 \ \Omega$$

Both the Thévenin equivalent voltage and impedance are shown in Fig. 3.14(d), and when the load is attached, as in Fig. 3.14(d), the current can be computed as

$$\hat{I} = \frac{\hat{V}_T}{20 + j15 - j35} = \frac{120 + j120}{20 - j20} = 0 + j6 \text{ A}$$

The Norton equivalent circuit is obtained via a simple voltage-to-current source transformation and is shown in Fig. 3.15. Here it is observed that a single current division gives

$$\hat{I} = \left[\frac{20 + j15}{20 + j15 - j35}\right](6.72 + j0.96) = 0 + j6$$
 A

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FIGURE 3.14 (a) A network in the phasor domain; (b) the network with the load removed; (c) the network for the computation of the Thévenin equivalent impedance; and (d) the Thévenin equivalent.



FIGURE 3.15 The Norton equivalent of Fig. 3.14(d).

Tellegen's Theorem

Tellegen's theorem states:

In an arbitrarily lumped network subject to KVL and KCL constraints, with reference directions of the branch currents and branch voltages associated with the KVL and KCL constraints, the product of all branch currents and branch voltages must equal zero.

Tellegen's theorem may be summarized by the equation

$$\sum_{k=1}^{b} v_k j_k = 0$$

where the lower case letters v and j represent instantaneous values of the branch voltages and branch currents, respectively, and where b is the total number of branches. A matrix representation employing the branch current and branch voltage vectors also exists. Because **V** and **J** are column vectors

$$\mathbf{V} \cdot \mathbf{J} = \mathbf{V}^T \mathbf{J} = \mathbf{J}^T \mathbf{V}$$

The prerequisite concerning the KVL and KCL constraints in the statement of Tellegen's theorem is of crucial importance.

Example 3.3. Figure 3.16 displays an oriented graph of a particular network in which there are six branches labeled with numbers within parentheses and four nodes labeled by numbers within circles. Several known branch currents and branch voltages are indicated. Because the type of elements or their values is not germane to the construction of the graph, the other branch currents and branch voltages may be evaluated from repeated applications of KCL and KVL. KCL may be used first at the various nodes.

node 3:
$$j_2 = j_6 - j_4 = 4 - 2 = 2$$
 A
node 1: $j_3 = -j_1 - j_2 = -8 - 2 = -10$ A
node 2: $j_5 = j_3 - j_4 = -10 - 2 = -12$ A

Then KVL gives

$$v_3 = v_2 - v_4 = 8 - 6 = 2 V$$

 $v_6 = v_5 - v_4 = -10 - 6 = -16 V$
 $v_1 = v_2 + v_6 = 8 - 16 = -8 V$





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The transpose of the branch voltage and current vectors are

$$\mathbf{V}_T = \begin{bmatrix} -8 & 8 & 2 & 6 & -10 & -16 \end{bmatrix} \mathbf{V}$$

and

$$\mathbf{J}_T = \begin{bmatrix} 8 & 2 & -10 & 2 & -12 & 4 \end{bmatrix} \mathbf{V}$$

The scalar product of V and J gives

$$-8(8) + 8(2) + 2(-10) + 6(2) + (-10)(-12) + (-16)(4) = -148 + 148 = 0$$

and Tellegen's theorem is confirmed.

Maximum Power Transfer

The maximum power transfer theorem pertains to the connections of a load to the Thévenin equivalent of a source network in such a manner as to transfer maximum power to the load. For a given network operating at a prescribed voltage with a Thévenin equivalent impedance

$$Z_T = |Z_T| / \theta_T$$

the real power drawn by any load of impedance

$$Z_o = |Z_o| / \underline{\theta}_o$$

is a function of just two variables, $|Z_o|$ and θ_o . If the power is to be a maximum, there are three alternatives to the selection of $|Z_o|$ and θ_o :

(1) Both $|Z_o|$ and θ_o are at the designer's discretion and both are allowed to vary in any manner in order to achieve the desired result. In this case, the load should be selected to be the complex conjugate of the Thévenin equivalent impedance

$$Z_o = Z_T^*$$

(2) The angle, θ_o , is fixed but the magnitude, $|Z_o|$, is allowed to vary. For example, the analyst may select and fix $\theta_o = 0^\circ$. This requires that the load be resistive (Z is entirely real). In this case, the value of the load resistance should be selected to be equal to the magnitude of the Thévenin equivalent impedance

$$R_o = |Z_T|$$

(3) The magnitude of the load impedance, $|Z_o|$, can be fixed, but the impedance angle, θ_o , is allowed to vary. In this case, the value of the load impedance angle should be

$$\theta_o = \arcsin\left[-\frac{2|Z_o||Z_T|\sin\theta_T}{|Z_o|^2 + |Z_T|^2}\right]$$

Example 3.4. Figure 3.17(a) is identical to Fig. 3.14(b) with the exception of a load, Z_{o} substituted for the capacitive load. The Thévenin equivalent is shown in Fig. 3.17(b). The value of Z_{o} to transfer maximum power



FIGURE 3.17 (a) A network for which the load, Z_o , is to be selected for maximum power transfer, and (b) the Thévenin equivalent of the network.

is to be found if its elements are unrestricted, if it is to be a single resistor, or if the magnitude of Z_o must be 20 Ω but its angle is adjustable.

For maximum power transfer to Z_o when the elements of Z_o are completely at the discretion of the network designer, Z_o must be the complex conjugate of Z_T

$$Z_o = Z_T^* = 20 - j15 \ \Omega$$

If Z_o is to be a single resistor, R_o then the magnitude of $Z_o = R_o$ must be equal to the magnitude of Z_T . Here

$$Z_T = 20 + j15 = 25 / 36.87^\circ$$

so that

$$R_o = |Z_o| = 25 \ \Omega$$

If the magnitude of Z_o must be 20 Ω but the angle is adjustable, the required angle is calculated from

$$\theta_o = \arcsin\left[-\frac{2 |Z_o| |Z_T|}{|Z_o|^2 + |Z_T|^2} \sin \theta_T\right]$$

= $\arcsin\left[-\frac{2(20)(25)}{(20)^2 + (25)^2} \sin \frac{/36.87^\circ}{36.87^\circ}\right]$
= $\arcsin(-0.585) = -35.83^\circ$

This makes Z_o

$$Z_o = 20 \ /-35.83^\circ = 16.22 - j11.71 \ \Omega$$

The Reciprocity Theorem

The reciprocity theorem is a useful general theorem that applies to all linear, passive, and bilateral networks. However, it applies only to cases where current and voltage are involved.

The ratio of a single excitation applied at one point to an observed response at another is invariant with respect to an interchange of the points of excitation and observation.

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FIGURE 3.18 Two networks which can be used to illustrate the reciprocity principle.

The reciprocity principle also applies if the excitation is a current and the observed response is a voltage. It will not apply, in general, for voltage–voltage and current–current situations, and, of course, it is not applicable to network models of nonlinear devices.

Example 3.5. It is easily shown that the positions of v_s and i in Fig. 3.18(a) may be interchanged as in Fig. 3.18(b) without changing the value of the current i.

In Fig. 3.18(a), the resistance presented to the voltage source is

$$R = 4 + \frac{3(6)}{3+6} = 4 + 2 = 6 \Omega$$

Then

$$i_a = \frac{v_s}{R} = \frac{36}{6} = 6 \text{ A}$$

and by current division

$$i_a = \frac{6}{6+3}i_a = \left(\frac{2}{3}\right)6 = 4$$
 A

In Fig. 3.18(b), the resistance presented to the voltage source is

$$R = 3 + \frac{6(4)}{6+4} = 3 + \frac{12}{5} = \frac{27}{5} \Omega$$

Then

$$i_b = \frac{v_s}{R} = \frac{36}{27/5} = \frac{180}{27} = \frac{20}{3}$$
 A

and again, by current division

$$i = \frac{6}{4+6}i_b = \left(\frac{3}{5}\right)\frac{20}{3} = 4$$
 A

The network is reciprocal.

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The Substitution and Compensation Theorems

The Substitution Theorem

Any branch in a network with branch voltage, v_k , and branch current, i_k , can be replaced by another branch provided it also has branch voltage, v_k , and branch current, i_k .

The Compensation Theorem

In a linear network, if the impedance of a branch carrying a current \hat{I} is changed from Z to $Z + \Delta Z$, then the corresponding change of any voltage or current elsewhere in the network will be due to a compensating voltage source, $\Delta Z \hat{I}$, placed in series with $Z + \Delta Z$ with polarity such that the source, $\Delta Z \hat{I}$, is opposing the current \hat{I} .

Defining Terms

- **Linear network:** A network in which the parameters of resistance, inductance, and capacitance are constant with respect to voltage or current or the rate of change of voltage or current and in which the voltage or current of sources is either independent of or proportional to other voltages or currents, or their derivatives.
- **Maximum power transfer theorem:** In any electrical network which carries direct or alternating current, the maximum possible power transferred from one section to another occurs when the impedance of the section acting as the load is the complex conjugate of the impedance of the section that acts as the source. Here, both impedances are measured across the pair of terminals in which the power is transferred with the other part of the network disconnected.
- **Norton theorem:** The voltage across an element that is connected to two terminals of a linear, bilateral network is equal to the short-circuit current between these terminals in the absence of the element, divided by the admittance of the network looking back from the terminals into the network, with all generators replaced by their internal admittances.
- **Principle of superposition:** In a linear electrical network, the voltage or current in any element resulting from several sources acting together is the sum of the voltages or currents from each source acting alone.
- **Reciprocity theorem:** In a network consisting of linear, passive impedances, the ratio of the voltage introduced into any branch to the current in any other branch is equal in magnitude and phase to the ratio that results if the positions of the voltage and current are interchanged.
- **Thévenin theorem:** The current flowing in any impedance connected to two terminals of a linear, bilateral network containing generators is equal to the current flowing in the same impedance when it is connected to a voltage generator whose voltage is the voltage at the open-circuited terminals in question and whose series impedance is the impedance of the network looking back from the terminals into the network, with all generators replaced by their internal impedances.

Related Topics

2.2 Ideal and Practical Sources • 3.4 Power and Energy

References

J. D. Irwin, *Basic Engineering Circuit Analysis*, 4th ed., New York: Macmillan, 1993.
A. D. Kraus, *Circuit Analysis*, St. Paul: West Publishing, 1991.
J. W. Nilsson, *Electric Circuits*, 5th ed., Reading, Mass.: Addison-Wesley, 1995.

Further Information

Three texts listed in the References have achieved widespread usage and contain more details on the material contained in this section.

3.4 Power and Energy

Norman Balabanian and Theodore A. Bickart

The concept of the voltage v between two points was introduced in Section 3.1 as the energy w expended per unit charge in moving the charge between the two points. Coupled with the definition of current i as the time rate of charge motion and that of power p as the time rate of change of energy, this leads to the following fundamental relationship between the power delivered to a two-terminal electrical component and the voltage and current of that component, with standard references (meaning that the voltage reference plus is at the tail of the current reference arrow) as shown in Fig. 3.19:



FIGURE 3.19 Power delivered to a circuit.

$$p = vi \tag{3.1}$$

Assuming that the voltage and current are in volts and amperes, respectively, the power is in *watts*. This relationship applies to any two-terminal component or network, whether linear or nonlinear.

The power delivered to the basic linear resistive, inductive, and capacitive elements is obtained by inserting the v-i relationships into this expression. Then, using the relationship between power and energy (power as the time derivative of energy and energy, therefore, as the integral of power), the energy stored in the capacitor and inductor is also obtained:

where the origin of time (t = 0) is chosen as the time when the capacitor voltage (respectively, the inductor current) is zero.

Tellegen's Theorem

A result that has far-reaching consequences in electrical engineering is Tellegen's theorem. It will be stated in terms of the networks shown in Fig. 3.20. These two are said to be topologically equivalent; that is, they are represented by the same graph but the components that constitute the branches of the graph are not necessarily the same in the two networks. they can even be nonlinear, as illustrated by the diode in one of the networks. Assuming all branches have standard references, including the source branches, Tellegen's theorem states that

$$\dot{\sum}_{\text{all } j} v_{bj} i_{aj} = 0$$

$$\mathbf{v}_{b}' \mathbf{i}_{a} = 0$$
(3.3)

In the second line, the variables are vectors and the prime stands for the transpose. The *a* and *b* subscripts refer to the two networks.

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FIGURE 3.20 Topologically equivalent networks.

This is an amazing result. It can be easily proved with the use of Kirchhoff's two laws.¹ The products of ν and *i* are reminiscent of power as in Eq. (3.1). However, the product of the voltage of a branch in one network and the current of its topologically corresponding branch (which may not even be the same type of component) in another network does not constitute power in either branch. furthermore, the variables in one network might be functions of time, while those of the other network might be steady-state phasors or Laplace transforms.

Nevertheless, some conclusions about power can be derived from Tellegen's theorem. Since a network is topologically equivalent to itself, the b network can be the same as the a network. In that case each vi product in Eq. (3.3) represents the power delivered to the corresponding branch, including the sources. The equation then says that if we add the power delivered to all the branches of a network, the result will be zero.

This result can be recast if the sources are separated from the other branches and one of the references of each source (current reference for each v-source and voltage reference for each *i*-source) is reversed. Then the vi product for each source, with new references, will enter Eq. (3.3) with a negative sign and will represent the power supplied by this source. When these terms are transposed to the right side of the equation, their signs are changed. The new equation will state in mathematical form that

In any electrical network, the sum of the power supplied by the sources is equal to the sum of the power delivered to all the nonsource branches.

This is not very surprising since it is equivalent to the law of conservation of energy, a fundamental principle of science.

AC Steady-State Power

Let us now consider the ac steady-state case, where all voltages and currents are sinusoidal. Thus, in the twoterminal circuit of Fig. 3.19:

$$v(t) = \sqrt{2} |V| \cos(\omega t + \alpha) \leftrightarrow V = |V| e^{j\alpha}$$

$$i(t) = \sqrt{2} |I| \cos(\omega t + \beta) \leftrightarrow I = |I| e^{j\beta}$$
(3.4)

The capital *V* and *I* are phasors representing the voltage and current, and their magnitudes are the corresponding rms values. The power delivered to the network at any instant of time is given by:

$$p(t) = v(t)i(t) = 2 |V| |I| \cos(\omega t + \alpha) \cos(\omega t + \beta)$$

= $[|V||I| \cos(\alpha - \beta)] + [|V||I| \cos(2\omega t + \alpha + \beta)]$ (3.5)

The last form is obtained by using trigonometric identities for the sum and difference of two angles. Whereas both the voltage and the current are sinusoidal, the instantaneous power contains a constant term (independent

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¹See, for example, N. Balabanian and T. A. Bickart, *Linear Network Theory*, Matrix Publishers, Chesterland, Ohio, 1981, chap. 9.



FIGURE 3.21 Instantaneous voltage, current, and power.

of time) in addition to a sinusoidal term. Furthermore, the frequency of the sinusoidal term is twice that of the voltage or current. Plots of v, i, and p are shown in Fig. 3.21 for specific values of α and β . The power is sometimes positive, sometimes negative. This means that power is sometimes delivered to the terminals and sometimes extracted from them.

The energy which is transmitted into the network over some interval of time is found by integrating the power over this interval. If the area under the positive part of the power curve were the same as the area under the negative part, the net energy transmitted over one cycle would be zero. For the values of α and β used in the figure, however, the positive area is greater, so there is a net transmission of energy toward the network. The energy flows back from the network to the source over part of the cycle, but on the average, more energy flows toward the network than away from it.

In Terms of RMS Values and Phase Difference

Consider the question from another point of view. The preceding equation shows the power to consist of a constant term and a sinusoid. The average value of a sinusoid is zero, so this term will contribute nothing to the net energy transmitted. Only the constant term will contribute. This constant term is the average value of the power, as can be seen either from the preceding figure or by integrating the preceding equation over one cycle. Denoting the average power by *P* and letting $\theta = \alpha - \beta$, which is the angle of the network impedance, the average power becomes:

$$P = |V| |I| \cos \theta$$

= $|V| |I| \operatorname{Re}(e^{j\theta}) = \operatorname{Re}[|V|| |I| e^{j(\alpha - \beta)}]$
= $\operatorname{Re}[(|V|e^{j\alpha}) (|I| e^{-j\beta})]$
= $\operatorname{Re}(VI^*)$ (3.6)

The third line is obtained by breaking up the exponential in the previous line by the law of exponents. The first factor between square brackets in this line is identified as the phasor voltage and the second factor as the conjugate of the phasor current. The last line then follows. It expresses the average power in terms of the voltage and current phasors and is sometimes more convenient to use.

Complex and Reactive Power

The average ac power is found to be the real part of a complex quantity VI*, labeled S, that in rectangular form is

$$S = VI^{*} = |V| |I| e^{j\theta} = |V| |I| \cos \theta + j |V| |I| \sin \theta$$

= P + jQ (3.7)

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FIGURE 3.22 In-phase and quadrature components of V and I.

where

$$P = |V| |I| \cos \theta \quad (a)$$

$$Q = |V| |I| \sin \theta \quad (b)$$

$$S| = |V| |I| \quad (c)$$
(3.8)

We already know *P* to be the average power. Since it is the real part of some complex quantity, it would be reasonable to call it the **real power**. The complex quantity *S* of which *P* is the real part is, therefore, called the *complex power*. Its magnitude is the product of the rms values of voltage and current: |S| = |V| |I|. It is called the *apparent power* and its unit is the volt-ampere (VA). To be consistent, then we should call *Q* the *imaginary power*. This is not usually done, however; instead, *Q* is called the **reactive power** and its unit is a VAR (volt-ampere reactive).

Phasor and Power Diagrams

An interpretation useful for clarifying and understanding the preceding relationships and for the calculation of power is a graphical approach. Figure 3.22(a) is a phasor diagram of *V* and *I* in a particular case. The phasor voltage can be resolved into two components, one parallel to the phasor current (or in phase with *I*) and another perpendicular to the current (or in quadrature with it). This is illustrated in Fig. 3.22(b). Hence, the average power *P* is the magnitude of phasor *I* multiplied by the in-phase component of *V*; the reactive power *Q* is the magnitude of *I* multiplied by the quadrature component of *V*.

Alternatively, one can imagine resolving phasor I into two components, one in phase with V and one in quadrature with it, as illustrated in Fig. 3.22(c). Then P is the product of the magnitude of V with the in-phase component of I, and Q is the product of the magnitude of V with the quadrature component of I. Real power is produced only by the in-phase components of V and I. The quadrature components contribute only to the reactive power.

The in-phase or quadrature components of *V* and *I* do not depend on the specific values of the angles of each, but on their phase difference. One can imagine the two phasors in the preceding diagram to be rigidly held together and rotated around the origin by any angle. As long as the angle θ is held fixed, all of the discussion of this section will still apply. It is common to take the current phasor as the reference for angle; that is, to choose $\beta = 0$ so that phasor *I* lies along the real axis. Then $\theta = \alpha$.

Power Factor

For any given circuit it is useful to know what part of the total complex power is real (average) power and what part is reactive power. This is usually expressed in terms of the **power factor** F_{p} , defined as the ratio of real power to apparent power:

Power factor
$$\doteq F_p = \frac{P}{|S|} = \frac{P}{|V||I|}$$
(3.9)

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FIGURE 3.23 Power waveform for unity and zero power factors.

Not counting the right side, this is a general relationship, although we developed it here for sinusoidal excitations. With $P = |V| |I| \cos \theta$, we find that the power factor is simply $\cos \theta$. Because of this, θ itself is called the power factor angle.

Since the cosine is an even function $[\cos(-\theta) = \cos \theta]$, specifying the power factor does not reveal the sign of θ . Remember that θ is the angle of the impedance. If θ is positive, this means that the current lags the voltage; we say that the power factor is a *lagging* power factor. On the other hand, if θ is negative, the current leads the voltage and we say this represent a *leading* power factor.

The power factor will reach its maximum value, unity, when the voltage and current are in phase. This will happen in a purely resistive circuit, of course. It will also happen in more general circuits for specific element values and a specific frequency.

We can now obtain a physical interpretation for the reactive power. When the power factor is unity, the voltage and current are in phase and $\sin \theta = 0$. Hence, the reactive power is zero. In this case, the instantaneous power is never negative. This case is illustrated by the current, voltage, and power waveforms in Fig. 3.23; the power curve never dips below the axis, and there is no exchange of energy between the source and the circuit. At the other extreme, when the power factor is zero, the voltage and current are 90° out of phase and $\sin \theta = 1$. Now the reactive power is a maximum and the average power is zero. In this case, the instantaneous power is positive over half a cycle (of the voltage) and negative over the other half. All the energy delivered by the source over half a cycle is returned to the source by the circuit over the other half.

It is clear, then, that the reactive power is a measure of the exchange of energy between the source and the circuit without being used by the circuit. Although none of this exchanged energy is dissipated by or stored in the circuit, and it is returned unused to the source, nevertheless it is temporarily made available to the circuit by the source.¹

Average Stored Energy

The average ac energy stored in an inductor or a capacitor can be established by using the expressions for the instantaneous stored energy for arbitrary time functions in Eq. (3.2), specifying the time function to be sinusoidal, and taking the average value of the result.

$$W_L = \frac{1}{2} L |I|^2 \qquad W_C = \frac{1}{2} C |V|^2$$
 (3.10)

¹Power companies charge their industrial customers not only for the average power they use but for the reactive power they return. There is a reason for this. Suppose a given power system is to deliver a fixed amount of average power at a constant voltage amplitude. Since $P = |V||I| \cos \theta$, the current will be inversely proportional to the power factor. If the reactive power is high, the power factor will be low and a high current will be required to deliver the given power. To carry a large current, the conductors carrying it to the customer must be correspondingly larger and better insulated, which means a larger capital investment in physical plant and facilities. It may be cost effective for customers to try to reduce the reactive power they require, even if they have to buy additional equipment to do so.

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FIGURE 3.24 A linear circuit delivering power to a load in the steady state.

Application of Tellegen's Theorem to Complex Power

An example of two topologically equivalent networks was shown in Fig. 3.20. Let us now specify that two such networks are linear, all sources are same-frequency sinusoids, they are operating in the steady state, and all variables are phasors. Furthermore, suppose the two networks are the same, except that the sources of network b have phasors that are the complex conjugates of those of network a. Then, if **V** and **I** denote the vectors of branch voltages and currents of network a, Tellegen's theorem in Eq. (3.3) becomes:

$$\sum_{\text{all}\,j}^{\star} V_{j}^{\star} I_{j} = V^{\star} I = 0$$
(3.11)

where V^* is the conjugate transpose of vector V.

This result states that the sum of the complex power delivered to all branches of a linear circuit operating in the ac steady state is zero. Alternatively stated, the total complex power delivered to a network by its sources equals the sum of the complex power delivered to its nonsource branches. Again, this result is not surprising. Since, if a complex quantity is zero, both the real and imaginary parts must be zero, the same result can be stated for the average power and for the reactive power.

Maximum Power Transfer

The diagram in Fig. 3.24 illustrates a two-terminal linear circuit at whose terminals an impedance Z_L is connected. The circuit is assumed to be operating in the ac steady state. The problem to be addressed is this: given the two-terminal circuit, how can the impedance connected to it be adjusted so that the maximum possible average power is transferred from the circuit to the impedance?

The first step is to replace the circuit by its Thévenin equivalent, as shown in Fig. 3.24(b). The current phasor in this circuit is $I = V_T/(Z_T + Z_L)$. The average power transferred by the circuit to the impedance is:

$$P = |I|^{2} \operatorname{Re}(Z_{L}) = \frac{|V_{T}|^{2} \operatorname{Re}(Z_{L})}{|Z_{T} + Z_{L}|^{2}}$$

$$= \frac{|V_{T}|^{2} R_{L}}{(R_{T} + R_{L})^{2} + (X_{T} + X_{L})^{2}}$$
(3.12)

In this expression, only the load (that is, R_L and X_L) can be varied. The preceding equation, then, expresses a dependent variable (*P*) in terms of two independent ones (R_L and X_L).

What is required is to maximize *P*. For a function of more than one variable, this is done by setting the partial derivatives with respect to each of the independent variables equal to zero; that is, $\partial P/\partial R_L = 0$ and $\partial P/\partial X_L = 0$. Carrying out these differentiations leads to the result that maximum power will be transferred when the load impedance is the conjugate of the Thévenin impedance of the circuit: $Z_L = Z_T^*$. If the Thévenin impedance is purely resistive, then the load resistance must equal the Thévenin resistance.

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In some cases, both the load impedance and the Thévenin impedance of the source may be fixed. In such a case, the matching for maximum power transfer can be achieved by using a transformer, as illustrated in Fig. 3.25, where the impedances are both resistive. The transformer is assumed to be ideal, with turns ratio *n*. Maximum power is transferred if $n^2 = R_T/R_L$.



Measuring AC Power and Energy

FIGURE 3.25 Matching with an ideal transformer.

With ac steady-state average power given in the first line of Eq. (3.6), measuring the average power requires measuring the rms values of voltage and current, as well as the power factor. This is accomplished by the arrangement shown in Fig. 3.26, which includes a breakout of an electrodynamometer-type wattmeter. The current in the high-resistance pivoted coil is proportional to the voltage across the load. The current to the load and the pivoted coil together through the energizing coil of the electromagnet establishes a proportional to the product of the magnetic field strength and the current in the pivoted coil is proportional to the torque on the pivoted coil is proportional to the pivoted coil. If the current in the pivoted coil is negligible compared to that in the load, then the torque becomes essentially proportional to the product of the the energizing coil of the electromagnet). The dynamics of the pivoted coil together with the restraining spring, at ac power frequencies, ensures that the angular displacement of the pivoted coil becomes proportional to the average of the torque or, equivalently, the average power.

One of the most ubiquitous of electrical instruments is the induction-type watthour meter, which measures the energy delivered to a load. Every customer of an electrical utility has one, for example. In this instance the pivoted coil is replaced by a rotating conducting (usually aluminum) disk as shown in Fig. 3.27. An induced eddy current in the disk replaces the pivoted coil current interaction with the load-current-established magnetic field. After compensating for the less-than-ideal nature of the electrical elements making up the meter as just described, the result is that the disk rotates at a rate proportional to the average power to the load and the rotational count is proportional to the energy delivered to the load.

At frequencies above the ac power frequencies and, in some instances, at the ac power frequencies, electronic instruments are available to measure power and energy. They are not a cost-effective substitute for these meters in the monitoring of power and energy delivered to most of the millions upon millions of homes and businesses.

Defining Terms

AC steady-state power: Consider an ac source connected at a pair of terminals to an otherwise isolated network. Let $\sqrt{2} \cdot |V|$ and $\sqrt{2} \cdot |I|$ denote the peak values, respectively, of the ac steady-state voltage and current at the terminals. Furthermore, let θ denote the phase angle by which the voltage leads the current. Then the average power delivered by the source to the network would be expressed as $P = |V| \cdot |I| \cos(\theta)$.



FIGURE 3.26 A wattmeter connected to a load.



FIGURE 3.27 A watthour meter connected to a load.

- **Power and energy:** Consider an electrical source connected at a pair of terminals to an otherwise isolated network. Power, denoted by p, is the time rate of change in the energy delivered to the network by the source. This can be expressed as p = vi, where v, the voltage across the terminals, is the energy expended per unit charge in moving the charge between the pair of terminals and i, the current through the terminals, is the time rate of charge motion.
- **Power factor:** Consider an ac source connected at a pair of terminals to an otherwise isolated network. The power factor, the ratio of the real power to the apparent power $|V| \cdot |I|$, is easily established to be $\cos(\theta)$, where θ is the power factor angle.
- **Reactive power:** Consider an ac source connected at a pair of terminals to an otherwise isolated network. The reactive power is a measure of the energy exchanged between the source and the network without being dissipated in the network. The reactive power delivered would be expressed as $Q = |V| \cdot |I| \sin(\theta)$.
- **Real power:** Consider an ac source connected at a pair of terminals to an otherwise isolated network. The real power, equal to the average power, is the power dissipated by the source in the network.
- **Tellegen's theorem:** Two networks, here including all sources, are topologically equivalent if they are similar structurally, component by component. Tellegen's theorem states that the sum over all products of the product of the current of a component of ne network, network *a*, and of the voltage of the corresponding component of the other network, network *b*, is zero. This would be expressed as $\sum_{\text{all } j} v_{bj} i_{aj} = 0$. From this general relationship it follows that in any electrical network, the sum of the power supplied by the sources is equal to the sum of the power delivered to all the nonsource components.

Related Topic

3.3 Network Theorems

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3.5 Three-Phase Circuits

Norman Balabanian

Figure 3.28(a) represents the basic circuit for considering the flow of power from a single sinusoidal source to a load. The power can be thought to cross an imaginary boundary surface (represented by the dotted line in the figure) separating the source from the load. Suppose that:

$$v(t) = \sqrt{2} |V| \cos(\omega t + \alpha)$$

$$i(t) = \sqrt{2} |I| \cos(\omega t + \beta)$$
(3.13)

Then the power to the load at any instant of time is

$$p(t) = |V||I| [\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)]$$
(3.14)

The instantaneous power has a constant term and a sinusoidal term at twice the frequency. The quantity in brackets fluctuates between a minimum value of $\cos(\alpha - \beta) - 1$ and a maximum value of $\cos(\alpha - \beta) + 1$. This fluctuation of power delivered to the load has certain disadvantages in some situations where the transmission of power is the purpose of a system. An electric motor, for example, operates by receiving electric power and transmitting mechanical (rotational) power at its shaft. If the electric power is delivered to the motor in spurts, the motor is likely to vibrate. In order to run satisfactorily, a physically larger motor will be needed, with a larger shaft and flywheel, to provide inertia than would be the case if the delivered power were constant.

This problem is overcome in practice by the use of what is called a *three-phase* system. This section will provide a brief discussion of three-phase systems.

Consider the circuit in Fig. 3.28(b). This arrangement is similar to a combination of three of the simple circuits in Fig. 3.28(a) connected in such a way that each one shares the return connection from O to N. The three sources can be viewed collectively as a single source and the three loads—which are assumed to be



FIGURE 3.28 Flow of power from source to load.

identical—can be viewed collectively as a single load. Then, as before, the dotted line represents a surface separating the source from the load. Each of the individual sources and loads is referred to as one *phase* of the three-phase system.

The three sources are assumed to have the same frequency; they are said to be *synchronized*. It is also assumed that the three voltages have the same rms values and the phase difference between each pair of voltages is $\pm 120^{\circ}$ ($2\pi/3$ rad). Thus, they can be written:

$$v_{a} = \sqrt{2} |V| \cos(\omega t + \alpha_{1}) \quad \leftrightarrow \qquad V_{a} = |V| e^{j0^{\circ}}$$

$$v_{b} = \sqrt{2} |V| \cos(\omega t + \alpha_{2}) \quad \leftrightarrow \qquad V_{b} = |V| e^{-j120^{\circ}} \qquad (3.15)$$

$$v_{c} = \sqrt{2} |V| \cos(\omega t + \alpha_{3}) \quad \leftrightarrow \qquad V_{c} = |V| e^{j120^{\circ}}$$

The **phasors** representing the sinusoids have also been shown. For convenience, the angle of v_a has been chosen as the reference for angles; v_b lags v_a by 120° and v_c leads v_a by 120°.

Because the loads are identical, the rms values of the three currents shown in the figure will also be the same and the phase difference between each pair of them will be $\pm 120^{\circ}$. Thus, the currents can be written:

$$i_{1} = \sqrt{2} |I| \cos(\omega t + \beta_{1}) \iff I_{1} = |I| e^{j\beta_{1}}$$

$$i_{2} = \sqrt{2} |I| \cos(\omega t + \beta_{2}) \iff I_{2} = |I| e^{j(\beta_{1} - 120^{\circ})}$$

$$i_{3} = \sqrt{2} |I| \cos(\omega t + \beta_{3}) \iff I_{3} = |I| e^{j(\beta_{1} + 120^{\circ})}$$
(3.16)

Perhaps a better form for visualizing the voltages and currents is a graphical one. Phasor diagrams for the voltages separately and the currents separately are shown in Fig. 3.29. The value of angle β_1 will depend on the load. An interesting result is clear from these diagrams. First, V_2 and V_3 are each other's conjugates. So if we add them, the imaginary parts cancel and the sum will be real, as illustrated by the construction in the voltage diagram. Furthermore, the construction shows that this real part is negative and equal in size to V_1 . Hence, the sum of the three voltages is zero. The same is true of the sum of the three currents, as can be established graphically by a similar construction.



FIGURE 3.29 Voltage and current phasor diagrams.



FIGURE 3.30 Wye-connected three-phase system.

By Kirchhoff's current law applied at node N in Fig. 3.28(b), we find that the current in the return line is the sum of the three currents in Eq. (3.16). However, since this sum was found to be zero, the return line carries no current. Hence it can be removed entirely without affecting the operation of the system. The resulting circuit is redrawn in Fig. 3.30. Because of its geometrical form, this connection of both the sources and the loads is said to be a wye (Y) connection.

The instantaneous power delivered by each of the sources has the form given in Eq. (3.14), consisting of a constant term representing the average power and a double-frequency sinusoidal term. The latter, being sinusoidal, can be represented by a phasor also. The only caution is that a different frequency is involved here, so this power phasor should not be mixed with the voltage and current phasors in the same diagram or calculations. Let |S| = |V||I| be the apparent power delivered by each of the three sources and let the three power phasors be S_a , S_b , and S_c , respectively. Then:

$$S_{a} = |S| e^{j(\alpha_{1} + \beta_{1})} = |S| e^{j\beta_{1}}$$

$$S_{b} = |S| e^{j(\alpha_{2} + \beta_{2})} = |S| e^{j(-120^{\circ} + \beta_{1} - 120^{\circ})} = |S| e^{j(\beta_{1} + 120^{\circ})}$$

$$S_{c} = |S| e^{j(\alpha_{3} + \beta_{3})} = |S| e^{j(+120^{\circ} + \beta_{1} + 120^{\circ})} = |S| e^{j(\beta_{1} - 120^{\circ})}$$
(3.17)

It is evident that the phase relationships among these three phasors are the same as the ones among the voltages and the currents. That is, the second leads the first by 120° and the third lags the first by 120°. Hence, just like the voltages and the currents, the sum of these three phasors will also be zero. This is a very significant result. Although the instantaneous power delivered by each source has a constant component and a sinusoidal component, when the three powers are added, the sinusoidal components add to zero, leaving only the constants. Thus, the total power delivered to the three loads is constant.

To determine the value of this constant power, use Eq. (3.14) as a model. The contribution of the *k*th source to the total (constant) power is $|S| \cos(\alpha_k - \beta_k)$. One can easily verify that $\alpha_k - \beta_k = \alpha_1 - \beta_1 = -\beta_1$. The first equality follows from the relationships among the α 's from Eq. (3.15) and among the β 's from Eq. (3.16). The choice of $\alpha_1 = 0$ leads to the last equality. Hence, the constant terms contributed to the power by each source are the same. If *P* is the total average power, then:

$$P = P_a + P_b + P_c = 3P_a = 3 |V||I| \cos(\alpha_1 - \beta_1)$$
(3.18)

Although the angle α_1 has been set equal to zero, for the sake of generality we have shown it explicitly in this equation.

What has just been described is a *balanced* three-phase three-wire power system. The three sources in practice are not three independent sources but consist of three different parts of the same generator. The same is true

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FIGURE 3.31 Three-phase circuit with nonzero winding and line impedances.

of the loads.¹ What has been described is ideal in a number of ways. First, the circuit can be *unbalanced*—for example, by the loads being somewhat unequal. Second, since the real devices whose ideal model is a voltage source are coils of wire, each source should be accompanied by a branch consisting of the coil inductance and resistance. Third, since the power station (or the distribution transformer at some intermediate point) may be at some distance from the load, the parameters of the physical line carrying the power (the line inductance and resistance) must also be inserted in series between the source and the load.

For an unbalanced system, the analysis of this section does not apply. An entirely new analytical technique is required to do full justice to such a system.² However, an understanding of balanced circuits is a prerequisite for tackling the unbalanced case.

The last two of the conditions that make the circuit less than ideal (line and source impedances) introduce algebraic complications, but nothing fundamental is changed in the preceding theory. If these two conditions are taken into account, the appropriate circuit takes the form shown in Fig. 3.31. Here the internal impedance of a source and the line impedance connecting that source to its load are both connected in series with the corresponding load. Thus, instead of the impedance in each phase being Z, it is $Z + Z_w + Z_l$, where w and l are subscripts standing for "winding" and "line," respectively. Hence, the rms value of each current is

$$|I| = \frac{|V|}{|Z + Z_w + Z_l|}$$
(3.19)

instead of |V|/|Z|. All other results we had arrived at remain unchanged. namely, that the sum of the phase currents is zero and that the sum of the phase powers is a constant. The detailed calculations simply become a little more complicated.

One other point, illustrated for the loads in Fig. 3.32, should be mentioned. Given wye-connected sources or loads, the wye and the **delta** can be made equivalent by proper selection of the arms of the delta. Thus,

¹An ac power generator consists of (a) a rotor, which produces a magnetic field and which is rotated by a prime mover (say a turbine), and (b) a stator on which are wound one or more coils of wire. In three-phase systems, the number of coils is three. The rotating magnetic field induces a voltage in each of the coils. The 120° leading and lagging phase relationships among these voltages are obtained by distributing the conductors of the coils around the circumference of the stator so that they are separated geometrically by 120°. Thus, the three sources described in the text are in reality a single physical device, a single generator. Similarly, the three loads might be the three windings on a three-phase motor, again a single physical device.

²The technique for analyzing unbalanced circuits utilizes what are called *symmetrical components*.

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FIGURE 3.32 Wye connection and delta connection.

either the sources in Fig. 3.30 or the loads, or both, can be replaced by a delta equivalent; thus we can conceive of four different three-phase circuits; wye-wye, delta-wye, wye-delta, and delta-delta. Not only can we conceive of them, they are extensively used in practice.

It is not worthwhile to carry out detailed calculations for these four cases. Once the basic properties described here are understood, one should be able to make the calculations. Observe, however, that in the delta structure, there is no neutral connection, so the phase voltages cannot be measured. The only voltages that can be measured are the *line-to-line* or simply the *line* voltages. These are the differences of the phase voltages taken in pairs, as is evident from Fig. 3.31.

Defining Terms

- **Delta connection:** The sources or loads in a three-phase system connected end-to-end, forming a closed path, like the Greek letter Δ .
- **Phasor:** A complex number representing a sinusoid; its magnitude and angle are the rms value and phase of the sinusoid, respectively.
- **Wye connection:** The three sources or loads in a three-phase system connected to have one common point, like the letter Y.

Related Topic

9.2 Three-Phase Connections

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3.6 Graph Theory¹

Shu-Park Chan

Topology is a branch of mathematics; it may be described as "the study of those properties of geometric forms that remain invariant under certain transformations, as bending, stretching, etc."² Network topology (or

¹Based on S.-P. Chan, "Graph theory and some of its applications in electrical network theory," in *Mathematical Aspects of Electrical Network Analysis*, vol. 3, *SIAM/AMS Proceedings*, American Mathematical Society, Providence, R.I., 1971. With permission.

²This brief description of topology is quoted directly from the *Random House Dictionary of the English Language*, Random House, New York, 1967.



FIGURE 3.33 A passive network N with a voltage driver E.

network graph theory) is a study of (electrical) networks in connection with their nonmetric geometrical (namely topological) properties by investigating the interconnections between the branches and the nodes of the networks. Such a study will lead to important results in network theory such as algorithms for formulating network equations and the proofs of various basic network theorems [Chan, 1969; Seshu and Reed, 1961].

The following are some basic definitions in network graph theory, which will be needed in the development of topological formulas in the analysis of linear networks and systems.

A linear graph (or simply a *graph*) is a set of line segments called *edges* and points called *vertices*, which are the endpoints of the edges, interconnected in such a way that the edges are connected to (or *incident* with) the vertices. The *degree* of a vertex of a graph is the number of edges incident with that vertex.

A subset G_i of the edges of a given graph G is called a **subgraph** of G. If G_i does not contain all of the edges of G, it is a **proper subgraph** of G. A **path** is a subgraph having all vertices of degree 2 except for the two endpoints, which are of degree 1 and are called the *terminals* of the path. The set of all edges in a path constitutes a **path-set**. If the two terminals of a path coincide, the path is a closed path and is called a **circuit** (or **loop**). The set of all edges contained in a circuit is called a **circuit-set** (or **loop-set**).

A graph or subgraph is said to be **connected** if there is at least one path between *every* pair of its vertices. A **tree** of a connected graph G is a connected subgraph which contains all the vertices of G but no circuits. The edges contained in a tree are called the **branches of the tree**. A 2-tree of a connected graph G is a (proper) subgraph of G consisting of two unconnected circuitless subgraphs, each subgraph itself being connected, which together contain all the vertices of G. Similarly, a k-tree is a subgraph of K unconnected circuitless subgraphs, each subgraph being connected, which together include all the vertices of G. The **k-tree admittance product of a k-tree** is the product of the admittances of all the branches of the k-tree.

Example 3.5. The graph *G* shown in Fig. 3.34 is the graph of the network *N* of Fig. 3.33. The edges of *G* are e_1 , e_2 , e_4 , e_5 , and e_6 ; the vertices of *G* are V_1 , V_2 , and V_3 , and V_4 . A path of *G* is the subgraph G_1 consisting of edges e_2 , e_3 , and e_6 with vertices V_2 and V_4 as terminals. Thus, the set $\{e_2, e_3, e_6\}$ is a path-set. With edge e_4 added to G_1 , we form another subgraph G_2 , which is a circuit since as far as G_2 is concerned all its vertices are of degree 2. Hence the set $\{e_2, e_3, e_4, e_6\}$ is a circuit-set. Obviously, *G* is a connected graph since there exists a path between every pair of vertices of *G*. A tree of *G* may be the subgraph consisting of edges e_1 , e_4 , and e_6 . Two other trees of *G* are $\{e_2, e_5, e_6\}$ and $\{e_3, e_4, e_5\}$. A 2-tree of *G* is $\{e_2, e_4\}$; another one is $\{e_3, e_6\}$; and still another one is $\{e_3, e_5\}$. Note that both $\{e_2, e_4\}$ and $\{e_3, e_6\}$ are subgraphs which obviously satisfy the definition of a 2-tree in the sense that each contains two disjoint circuitless connected subgraphs, both of which include all the four vertices of *G*. Thus, $\{e_3, e_5\}$ does not seem to be a 2-tree. However, if we



FIGURE 3.34 The graph *G* of the network *N* of Fig. 3.33.

agree to consider $\{e_3, e_5\}$ as a subgraph which contains edges e_3 and e_5 plus the isolated vertex V_4 , we see that $\{e_3, e_5\}$ will satisfy the definition of a 2-tree since it now has two circuitless connected subgraphs with e_3 and e_5 forming one of them and the vertex V_4 alone forming the other. Moreover, both subgraphs together indeed

include all the four vertices of *G*. It is worth noting that a 2-tree is obtained from a tree by removing *any one* of the branches from the tree; in general, a *k*-tree is obtained from a (k-1) tree by removing from it any one of its branches. Finally, the tree admittance product of the tree $\{e_2, e_5, e_6\}$ is $\frac{1}{2}$ $\frac{1}{5}$ $\frac{1}{6}$; the 2-tree admittance product of the tree $\{e_3, e_5\}$ is $\frac{1}{3}$ $\frac{1}{5}$ (with the admittance of a vertex defined to be 1).

The k-Tree Approach

The development of the analysis of passive electrical networks using topological concepts may be dated back to 1847 when Kirchhoff formulated his set of topological formulas in terms of resistances and the branch-current system of equations. In 1892, Maxwell developed another set of topological formulas based on the *k*-tree concept, which are the duals of Kirchhoff's. These two sets of formulas were supported mainly by heuristic reasoning and no formal proofs were then available.

In the following we shall discuss only Maxwell's topological formulas for linear networks without mutual inductances.

Consider a network *N* with *n* independent nodes as shown in Fig. 3.35. The node 1' is taken as reference (datum) node. the voltages V_1, V_2, \ldots, V_n (which are functions of *s*) are the



FIGURE 3.35 A network *N* with *n* independent nodes.

transforms of the node-pair voltages (or simply node voltages) $v_1, v_2, ..., v_n$ (which are function *s* of *t*) between the *n* nodes and the reference node 1' with the plus polarity marks at the *n* nodes. It can be shown [Aitken, 1956] that the matrix equation for the *n* independent nodes of *N* is given by

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix}$$
(3.20)

or, in abbreviated matrix notation,

$$Y_n V_n = I_n \tag{3.21}$$

where Y_n is the node admittance matrix, V_n the $n \times 1$ matrix of the node voltage transforms, and I_n the $n \times 1$ matrix of the transforms of the known current sources.

For a relaxed passive one-port (with zero initial conditions) shown in Fig. 3.36, the driving-point impedance function $Z_d(s)$ and its reciprocal, namely driving-point admittance function $Y_d(s)$, are given by

$$Z_d(s) = V_1 / I_1 = \Delta_{11} / \Delta$$

and

$$Y_d(s) = 1/Z_d(s) = \Delta/\Delta_{11}$$

respectively, where Δ is the determinant of the node admittance matrix Y_n , and Δ_{11} is the (1,1)-cofactor of Δ .

Similarly, for a passive reciprocal *RLC* two-port (Fig. 3.37), the open-circuit impedances and the short-circuit admittances are seen to be

$$z_{11} = \Delta_{11} / \Delta \tag{3.22a}$$

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FIGURE 3.36 The network *N* driven by a single current source.



FIGURE 3.37 A passive two-port.

$$z_{12} = z_{21} = (\Delta_{12} - \Delta_{12'})/\Delta$$
(3.22b)

$$z_{22} = (\Delta_{22} + \Delta_{2'2'} - 2\Delta_{22'})/\Delta$$
 (3.22c)

and

$$y_{11} = (\Delta_{22} + \Delta_{2'2'} - 2\Delta_{22'})/(\Delta_{1122} + \Delta_{112'2'} - 2\Delta_{1122'})$$
(3.23a)

$$y_{12} = y_{21} = \Delta_{12'} - \Delta_{12} / (\Delta_{1122} + \Delta_{112'2'} - 2\Delta_{1122'})$$
(3.23b)

$$y_{22} = \Delta_{11} / (\Delta_{1122} + \Delta_{112'2'} - 2\Delta_{1122'})$$
(3.23c)

respectively, where Δ_{ij} is the (i,j)-cofactor of Δ , and Δ_{ijkm} is the cofactor of Δ by deleting rows *i* and *k* and columns *j* and *m* from Δ [Aitken, 1956].

Expressions in terms of network determinants and cofactors for other network transfer functions are given by (Fig. 3.38):

$$z_{12} = \frac{V_2}{I_1} = \frac{\Delta_{12} - \Delta_{12'}}{\Delta}$$
 (transfer impedance function) (3.24a)

$$G_{12} = \frac{V_2}{V_1} = \frac{\Delta_{12} - \Delta_{12'}}{\Delta_{11}}$$
 (voltage-ratio transfer function) (3.24b)

$$Y_{12} = Y_L G_{12} = Y_L \left(\frac{\Delta_{12} - \Delta_{12'}}{\Delta_{11}}\right) \quad \text{(transfer admittance function)} \quad (3.24c)$$

$$\alpha_{12} = Y_1 Z_{12} = Y_L \left(\frac{\Delta_{12} - \Delta_{12'}}{\Delta}\right) \quad (\text{current-ratio transfer function}) \quad (3.24d)$$

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FIGURE 3.38 A loaded passive two-port.

The topological formulas for the various network functions of a passive one-port or two-port are derived from the following theorems which are stated without proof [Chan, 1969].

Theorem 3.1. Let N be a passive network without mutual inductances. The determinant Δ of the node admittance matrix Y_n is equal to the sum of all tree-admittances of N, where a tree-admittance product $t^{(i)}(y)$ is defined to be the product of the admittance of all the branches of the tree $T^{(i)}$. That is,

$$\Delta = \det Y_n = \sum_i T^{(i)}(y)$$
(3.25)

Theorem 3.2. Let Δ be the determinant of the node admittance matrix Y_n of a passive network N with n + 1 nodes and without mutual inductances. Also let the reference node be denoted by 1'. Then the (j,j)-cofactor Δ_{jj} of Δ is equal to the sum of all the 2-tree-admittance products $T_{2j,1'}(y)$ of N, each of which contains node j in one part and node 1' as the reference node) and without mutual inductances is given by

$$\Delta_{jj} = \sum_{k} T^{(k)}_{2_{j,l'}}(y)$$
(3.26)

where the summation is taken over all the 2-tree-admittance products of the form $T_{2il'}(y)$.

Theorem 3.3. The (i,j)-cofactor Δ_{ij} of Δ of a relaxed passive network N with n independent nodes (with node 1' as the reference node) and without mutual inductances is given by

$$\Delta_{ij} = \sum_{k} T_{2_{ij,i'}}^{(k)}(y)$$
(3.27)

where the summation is taken over all the 2-tree-admittance products of the form $T_{2ij,1'}(y)$ with each containing nodes *i* and *j* in one connected port and the reference node 1' in the other.

For example, the topological formulas for the driving-point function of a passive one-port can be readily obtained from Eqs. (3.25) and (3.26) in Theorems 3.1 and 3.2 as stated in the next theorem.

Theorem 3.4. With the same notation as in Theorems 3.1 and 3.2, the driving-point admittance $Y_d(s)$ and the driving-point impedance $Z_d(s)$ of a passive one-port containing no mutual inductances at terminals 1 and 1' are given by

$$Y_{d}(s) = \frac{\Delta}{\Delta_{11}} = \frac{\sum_{i} T^{(i)}(y)}{\sum_{k} T^{(k)}_{2_{1,1}}(y)} \quad \text{and} \quad Z_{d}(s) = \frac{\Delta_{11}}{\Delta} = \frac{\sum_{k} T^{(k)}_{2_{1,1}}(y)}{\sum_{i} T^{(i)}(y)} \quad (3.28)$$

respectively.

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For convenience we define the following shorthand notation:

$$(a)V(Y) \equiv \sum_{i} T^{(i)}(y) = \text{sum of all tree-admittance products, and}$$
$$(b)W_{j,r}(y) \equiv \sum_{k} T_{2j,r}(y) = \text{sum of all } 2\text{-tree-admittance products with node } j \qquad (3.29)$$
and the reference node *r* contained in different parts.

Thus Eq. (3.28) may be written as

$$Y_d(s) = V(Y)/W_{1,1'}(Y)$$
 and $Z_d(s) = W_{1,1'}(Y)/V(Y)$ (3.30)

In a two-port network *N*, there are four nodes to be specified, namely, 1 and 1' at the input port (1,1') and nodes 2 and 2' at the output port (2,2'), as illustrated in Fig. 3.38. However, for a 2-tree of the type $T_{2ij,1'}$, only three nodes have been used, thus leaving the fourth one unidentified.

With very little effort, it can be shown that, in general, the following relationship holds:

$$W_{ij,1'}(Y) = W_{ijk,1'}(Y) + W_{ij,k1'}(Y)$$

or simply

$$W_{ij,1'} = W_{ijk,1'} + W_{ij,k1'} \tag{3.31}$$

where *i*, *j*, *k*, and 1' are the four terminals of *N* with 1' denoting the datum (reference) node. The symbol $W_{ijk,1'}$ denotes the sum of all the 2-tree-admittance products, each containing nodes *i*, *j*, and *k* in one connected part and the reference node, 1', in the other.

We now state the next theorem.

Theorem 3.5. With the same hypothesis and notation as stated earlier in this section,

$$\Delta_{12} - \Delta_{12'} = W_{12,12'}(Y) - W_{12',1'2}(Y)$$
(3.32)

It is interesting to note that Eq. (3.32) is stated by Percival [1953] in the following descriptive fashion:

$$\Delta_{12} - \Delta_{12'} = W_{12,12'} - W_{12',1'2} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2' \\ 1' & 0 & 0 & 2' \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2' \\ 1' & 0 & 2' \end{pmatrix}$$

which illustrates the two types of 2-trees involved in the formula. Hence, we state the topological formulas for z_{11} , z_{12} , and z_{22} in the following theorem.

Theorem 3.6. With the same hypothesis and notation as stated earlier in this section

$$z_{11} = W_{1,1'}(Y)/V(Y)$$
(3.33a)

$$z_{12} = z_{21} = \{W_{12,1'2'}(Y) - W_{12',1'2}(Y)\}/V(Y)$$
(3.33b)

$$z_{22} = W_{2,2'}(Y)/V(Y)$$
(3.33c)

We shall now develop the topological expressions for the short-circuit admittance functions. Let us denote by $U_{a,b,c}(Y)$ the sum of all 3-tree admittance products of the form $T_{3_{a,b,c}}(Y)$ with identical subscripts in both

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FIGURE 3.39 The network *N* of Example 3.7.

symbols to represent the same specified distribution of vertices. Then, following arguments similar to those of Theorem 3.5, we readily see that

$$\Delta_{1122} = \sum_{i} T_{3_{i,2,1'}}^{(i)}(y) \equiv U_{1,2,1'}(Y)$$
(3.34a)

$$\Delta_{112'2'} = \sum_{j} T_{3_{1,2',1'}}^{(j)}(y) \equiv U_{1,2',1'}(Y)$$
(3.34b)

$$\Delta_{1122'} = \sum_{k} T_{3_{1,22',1'}}^{(k)}(y) \equiv U_{1,22',1'}(U)$$
(3.34c)

where 1,1',2,2' are the four terminals of the two-port with 1' denoting the reference node (Fig. 3.39). However, we note that in Eqs. (3.34a) and (3.34b) only three of the four terminals have been specified. We can therefore further expand $U_{1,2,1'}$ and $U_{1,2',1'}$ to obtain the following:

$$\Delta_{1122} + \Delta_{112'2'} - 2\Delta_{1122'} = U_{12',2,1'} + U_{1,2,1'2'} + U_{12,2',1'} + U_{1,2',1'2}$$
(3.35)

For convenience, we shall use the shorthand notation ΣU to denote the sum of the right of Eq. (3.35). Thus, we define

$$\sum U = U_{12',2,1'} + U_{1,2,1'2'} + U_{12,2',1'} + U_{1,2',1'2}$$
(3.36)

Hence, we obtain the topological formulas for the short-circuit admittances as stated in the following theorem. *Theorem 3.7.* The short-circuit admittance functions y_{11} , y_{12} , and y_{22} of a passive two-port network with no mutual inductances are given by

$$y_{11} = W_{2,2'} / \sum U$$
 (3.37a)

$$y_{12} = y_{21} = (W_{12',1'2} - W_{12,1'2'}) / \sum U$$
 (3.37b)

$$y_{22} = W_{1,1'} / \sum U$$
 (3.37c)

where ΣU is defined in Eq. (3.36) above.

Finally, following similar developments, other network functions are stated in Theorem 3.8.

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Theorem 3.8. With the same notation as before,

$$Z_{12}(s) = \frac{W_{12,1'2'} - W_{12',1'2}}{V}$$
(3.38a)

$$G_{12}(s) = \frac{W_{12,1'2'} - W_{12',1'2}}{W_{1,1'}}$$
(3.38b)

$$Y_{12}(s) = Y_L \frac{W_{12,1'2'} - W_{12',1'2}}{W_{1,1'}}$$
(3.38c)

$$\alpha_{12}(s) = Y_L \frac{W_{12,1'2'} - W_{12',1'2}}{V}$$
(3.38d)

The Flowgraph Approach

Mathematically speaking, a linear electrical network or, more generally, a linear system can be described by a set of simultaneous linear equations. Solutions to these equations can be obtained either by the method of successive substitutions (elimination theory), by the method of determinants (Cramer's rule), or by any of the topological techniques such as Maxwell's *k*-tree approach discussed in the preceding subsection and the flow-graph techniques represented by the works of Mason [1953, 1956], and Coates [1959].

Although the methods using algebraic manipulations can be amended and executed by a computer, they do not reveal the physical situations existing in the system. The flowgraph techniques, on the other hand, show intuitively the causal relationships between the variables of the system of interest and hence enable the network analyst to have an excellent physical insight into the problem.

In the following, two of the more well-known flowgraph techniques are discussed, namely, the **signal-flowgraph** technique devised by Mason and the method based on the flowgraph of Coates and recently modified by Chan and Bapna [1967].

A signal-flowgraph G_m of a system S of n independent linear (algebraic) equations in n unknowns

$$\sum_{j=1}^{n} a_{ij} x_{j} = b_{i} \qquad i = 1, 2, \dots, n \qquad (3.39)$$

is a graph with junction points called *nodes* which are connected by directed line segments called *branches* with signals traveling along the branches only in the direction described by the arrows of the branches. A signal x_k traveling along a branch between x_k and x_j is multiplied by the gain of the branches g_{kj} , so that a signal of $g_{kj}x_k$ is delivered at node x_j . An *input node* (*source*) is a node which contains only outgoing branches; an *output node* (*sink*) is a node which has only incoming branches. A *path* is a continuous unidirectional succession of branches, all of which are traveling in the same direction; a *forward path* is a path from the input node to the output node along which all nodes are encountered exactly once; and a *feedback path* (*loop*) is a closed path which originates from and terminates at the same node, and along which all other nodes are encountered exactly once (the trivial case is a *self-loop* which contains exactly one node and one branch). A *path gain* is the product of all the branch gains of the path; similarly, a loop gain is the product of all the branch gains of the branches in a loop.

The procedure for obtaining the Mason graph from a system of linear algebraic equations may be described in the following steps: a. Arrange all the equations of the system in such a way that the *j*th dependent (output) variable x_j in the *j*th equation is expressed explicitly in terms of the other variables. Thus, if the system under study is given by Eq. (3.39), namely

$$a_{11}x_{1} + a_{12}x_{2} + \dots + a_{1n}x_{n} = b_{1}$$

$$a_{21}x_{1} + a_{22}x_{2} + \dots + a_{2n}x_{n} = b_{2}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{n1}x_{1} + a_{n2}x_{2} + \dots + a_{nn}x_{n} = b_{n}$$
(3.40)

where b_1, b_2, \ldots, b_n are inputs (sources) and x_1, x_2, \ldots, x_n are outputs, the equations may be rewritten as

$$\begin{aligned} x_1 &= \frac{1}{a_{11}} b_1 - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3 - \dots - \frac{a_{1n}}{a_{11}} x_n \\ x_2 &= \frac{1}{a_{22}} b_2 - \frac{a_{21}}{a_{22}} x_1 - \frac{a_{23}}{a_{22}} x_3 - \dots - \frac{a_{2n}}{a_{22}} x_n \\ \vdots &\vdots &\vdots &\vdots &\vdots \\ x_n &= \frac{1}{a_{nn}} b_n - \frac{a_{n1}}{a_{nn}} x_1 - \frac{a_{n2}}{a_{nn}} x_2 - \dots - \frac{a_{n-1,n-1}}{a_{nn}} x_{n-1} \end{aligned}$$
(3.41)

- b. The number of input nodes in the flowgraph is equal to the number of nonzero b's. That is, each of the source nodes corresponds to a nonzero b_i .
- c. To each of the output nodes is associated one of the dependent variables x_1, x_2, \ldots, x_n
- d. The value of the variable represented by a node is equal to the sum of all the incoming signals.
- e. The value of the variable represented by any node is transmitted onto all branches leaving the node.

It is a simple matter to write the equations from the flowgraph since every node, except the source nodes of the graph, represents an equation, and the equation associated with node k, for example, is obtained by equating to x_k the sum of all incoming branch gains multiplied by the values of the variables from which these branches originate.

Mason's general gain formula is now stated in the following theorem.

Theorem 3.9. Let G be the overall graph gain and G_k be the gain of the kth forward path from the source to the sink. Then

$$G = \frac{1}{\Delta} \sum_{k} G_k \Delta_k \tag{3.42}$$

where

$$\Delta = 1 \sum_{m} p_{m1} + \sum_{m} p_{m2} - \sum_{m} p_{m3} + \dots + (-1)^{j} \sum_{m} p_{mj}$$

 p_{m1} = loop gain (the product of all the branch gains around a loop)

 p_{m2} = product of the loop gains of the *m*th set of two nontouching loops

 p_{m3} = product of the loop gains of the *m*th set of three nontouching loops, and in general

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- p_{mj} = product of the loop gains of the *m*th set of *j* nontouching loops
- Δ_k = the value of Δ for that subgraph of the graph obtained by removing the *k*th forward path along with those branches touching the path

Mason's signal-flowgraphs constitute a very useful graphical technique for the analysis of linear systems. This technique not only retains the intuitive character of the block diagrams but at the same time allows one to obtain the gain between an input node and an output node of a signal-flowgraph by inspection. However, the derivation of the gain formula [Eq. (3.42)] is by no means simple, and, more importantly, if more than one input is present in the system, the gain cannot be obtained directly; that is, the principle of superposition must be applied to determine the gain due to the presence of more than one input. Thus, by slight modification of the conventions involved in Mason's signal-flowgraph, Coates [1959] was able to introduce the so-called "flowgraphs" which are suitable for direct calculation of gain.

Recently, Chan and Bapna [1967] further modified Coates's flowgraphs and developed a simpler gain formula based on the modified graphs. The definitions and the gain formula based on the modified Coates graphs are presented in the following discussion.

The flowgraph G_l (called the *modified Coates graph*) of a system S of n independent linear equations in n unknowns

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, 2, \dots, n$$

is an oriented graph such that the variable x_j in *S* is represented by a *node* (also denoted by x_j) in G_b and the coefficient a_{ij} of the variable x_j in *S* by a *branch* with a branch gain a_{ij} connected between nodes x_i and x_j in G_l and directed from x_j to x_i . Furthermore, a *source node* is included in G_l such that for each constant b_k in *S* there is a node with gain b_k in G_l from node 1 to node s_k . Graph G_{l0} is the subgraph of G_l obtained by deleting the source node l and all the branches connected to it. Graph G_{ij} is the subgraph of G_l obtained by first removing all the outgoing branches from node x_j and then short-circuiting node 1 to node x_j . A *loop set l* is a subgraph of G_{l0} that contains all the nodes of G_{l0} with each node having exactly one incoming and one outgoing branch. The product *p* of the gains of all the branches in *l* is called a *loop-set product*. A 2-*loop-set I*₂ is a subgraph of G_{lj} containing all the nodes of G_{lj} with each node having exactly one incoming and one outgoing branch. The product p_2 of the gains of all the branches in l_2 is called a 2-*loop-set product*.

The modified Coates gain formula is now stated in the following theorem.

Theorem 3.10. In a system of *n* independent linear equations in *n* unknowns

$$a_{ii}x_i = b_i$$
 $i = 1, 2, ..., n$

the value of the variable x_i is given by

$$x_{j} = \sum_{(\text{all } p_{2})} (-1)^{N_{1_{2}}} p_{2} / \sum_{(\text{all } p_{2})} (-1)^{N_{1}} p$$
(3.43)

where N_{l_2} is the number of loops in a 2-loop-set l_2 and N_l is the number of loops in a loop set l.

Since both the Mason graph G_m and the modified Coates graph G_l are topological representations of a system of equations it is logical that certain interrelationships exist between the two graphs so that one can be transformed into the other. Such interrelationships have been noted [Chan, 1969], and the transformations are briefly stated as follows:

A. *Transformation of* G_m *into* G_l . Graph G_m can be transformed into an equivalent Coates graph G_l (representing an equivalent system of equations) by the following steps:

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- a. Subtract 1 from the gain of each existing self-loop.
- b. Add a self-loop with a gain of -1 to each branch devoid of self-loop.
- c. Multiply by $-b_k$ the gain of the branch at the *k*th source node b_k (k = 1, 2, ..., r, *r* being the number of source nodes) and then combine all the (*r*) nodes into one source node (now denoted by 1).
- B. Transformation of G_l into G_m . Graph G_l can be transformed into G_m by the following steps:
 - a. Add 1 to the gain of each existing self-loop.
 - b. Add a self-loop with a gain of 1 to each node devoid of self-loop except the source node l.
 - c. Break the source node l into *r* source nodes (*r* being the number of branches connected to the source node l before breaking), and identify the *r* new sources nodes by b_1, b_2, \ldots, b , with the gain of the corresponding *r* branches multiplied by $-1/b_1, -1/b_2, \ldots, -1/b_r$, respectively, so that the new gains of these branches are all equal to l, keeping the edge orientations unchanged.

The gain formulas of Mason and Coates are the classical ones in the theory of flowgraphs. From the systems viewpoint, the Mason technique provides an excellent physical insight as one can visualize the signal flow through the subgraphs (forward paths and feedback loops) of G_m . The graph reduction technique based on the Mason graph enables one to obtain the gain expression using a step-by-step approach and at the same time observe the cause-and-effect relationships in each step. However, since the Mason formula computes the ratio of a specified output over *one* particular input, the principle of superposition must be used in order to obtain the overall gain of the system if more than one input is present. The Coates formula, on the other hand, computes the output directly regardless of the number of inputs present in the system, but because of such a direct computation of a given output, the graph reduction rules of Mason cannot be applied to a Coates graph since the Coates graph is *not* based on the same cause-effect formulation of equations as Mason's.

The k-Tree Approach Versus the Flowgraph Approach

When a linear network is given, loop or node equations can be written from the network, and the analysis of the network can be accomplished by means of either Coates's or Mason's technique.

However, it has been shown [Chan, 1969] that if the Maxwell *k*-tree approach is employed in solving a linear network, the redundancy inherent either in the direct expansion of determinants or in the flowgraph techniques described above can be either completely eliminated for passive networks or greatly reduced for active networks. This point and others will be illustrated in the following example.

Example 3.7. Consider the network N as shown in Fig. 3.39. Let us determine the voltage gain, $G_{12} = V_0 / V_1$, using (1) Mason's method, (2) Coates's method, and (3) the *k*-tree method.

The two node equations for the network are given by

for node 2:
$$(Y_a + Y_b + Y_e)V_2 + (-Y_3)V_0 = Y_aV_i$$

for node 3: $(-Y_e)V_2 + (Y_c + Y_d + Y_e)V_0 = Y_cV_i$ (3.44)

where

$$Y_a = 1/Z_a, Y_b = 1/Z_b, Y_c = 1/Z_c, Y_d = 1/Z_d \text{ and } Y_e = 1/Z_e$$

(1) Mason's approach. Rewrite the system of two equations (3.44) as follows:

$$V_{2} = \left(\frac{Y_{a}}{Y_{a} + Y_{b} + Y_{e}}\right) V_{i} + \left(\frac{Y_{e}}{Y_{a} + Y_{b} + Y_{e}}\right) V_{0}$$

$$V_{0} = \left(\frac{Y_{c}}{Y_{c} + Y_{d} + Y_{e}}\right) V_{i} + \left(\frac{Y_{e}}{Y_{c} + Y_{d} + Y_{e}}\right) V_{2}$$
(3.45)

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FIGURE 3.40 The Mason graph of *N*.

or

$$V_2 = AV_i + BV_0 \quad V_0 = CV_i + DV_2$$
 (3.46)

where

$$A = \frac{Y_a}{Y_a + Y_b + Y_e} \qquad B = \frac{Y_e}{Y_a + Y_b + Y_e}$$
$$C = \frac{Y_c}{Y_c + Y_d + Y_e} \qquad D = \frac{Y_e}{Y_c + Y_d + Y_e}$$

The Mason graph of system (3.46) is shown in Fig. 3.40, and according to the Mason graph formula (3.42), we have

$$\Delta = 1 - BD$$

$$G_C = C \quad \Delta_C = 1$$

$$G_{AD} = AD \quad \Delta_{AD} = 1$$

and hence

$$G_{12} = \frac{V_0}{V_1} = \frac{1}{\Delta} \sum_k G_k \Delta_k = \frac{1}{1 - BD} (C + AD)$$
$$= \frac{Y_c / (Y_c + Y_d + Y_e) + Y_a / (Y_a + Y_b + Y_e) (Y_c + Y_d + Y_e)}{1 - Y_e^2 / (Y_a + Y_b + Y_e) (Y_c + Y_d + Y_e)}$$

Upon cancellation and rearrangement of terms

$$G_{12} = \frac{Y_a Y_c + Y_a Y_e + Y_b Y_c + Y_c Y_e}{Y_a Y_c + Y_a Y_d + Y_a Y_e + Y_b Y_c + Y_b Y_d + Y_b Y_e + Y_c Y_e + Y_d Y_e}$$
(3.47)

(2) *Coates's approach.* From (3.44) we obtain the Coates graphs G_l , G_{l0} , and G_{l3} as shown in Fig. 3.41(a), (b), and (c), respectively. The set of all loop-sets of G_{l0} is shown in Fig. 3.42, and the set of all 2-loop-sets of G_{l3} is shown in Fig. 3.43. Thus, by Eq. (3.43),

$$V_0 = \frac{\sum_{(\text{all } p_2)} (-1)^{N_l_2} p_2}{\sum_{(\text{all } p)} (-1)^{N_l} p} = \frac{(-1)^1 (-Y_e) (Y_a V_i) + (-1)^2 (Y_a + Y_b + Y_e) (Y_c V_i)}{(-1)^1 (-Y_e) (-Y_e) + (-1)^2 (Y_a + Y_b + Y_e) (Y_c + Y_d + Y_e)}$$

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FIGURE 3.41 The Coates graphs: (a) G_l , (b) G_{l0} , and (c) G_{l3} .

Or, after simplification, we find

$$V_{0} = \frac{(Y_{a}Y_{c} + Y_{a}Y_{e} + Y_{b}Y_{c} + Y_{c}Y_{e})V_{i}}{Y_{a}Y_{c} + Y_{a}Y_{d} + Y_{a}Y_{e} + Y_{b}Y_{c} + Y_{b}Y_{d} + Y_{b}Y_{e} + Y_{c}Y_{e} + Y_{d}Y_{e}}$$
(3.48)

which gives the same ratio V_0/V_i as Eq. (3.47).

(3) The k-tree approach. Recall that the gain formula for V_0/V_i using the k-tree approach is given [Chan, 1969] by

$$\frac{V_0}{V_i} = \frac{\Delta_{13}}{\Delta_{11}} = \frac{W_{13,R}}{W_{1,R}} = \frac{\sum \left(\text{all } 2\text{-tree admittance products with nodes 1 and 3 in one part} \\ \text{and the reference node } R \text{ in the other part of each of such } 2\text{-tree} \right)}{\sum \left(\text{all } 2\text{-tree admittance products with node 1 in one part and} \\ \text{the reference node } R \text{ in the other part of each of such } 2\text{-tree} \right)}$$
(3.49)

where Δ_{13} and Δ_{11} are cofactors of the determinant Δ of the node admittance matrix of the network. Furthermore, it is noted that the 2-trees corresponding to Δ_{ii} may be obtained by finding all the trees of the modified graph G_i , which is obtained from the graph G of the network by short-circuiting node i (i being any node other than



FIGURE 3.42 The set of all loop-sets of G_{l_0} .



FIGURE 3.43 The set of all 2-loop-sets of G_{l_3} .



FIGURE 3.44 (a) Graph G, and (b) the modified graph G_l of G.

R) to the reference node *R*, and that the 2-trees corresponding to Δ_{ij} can be found by taking all those 2-trees each of which is a tree of both G_i and G_j [Chan, 1969]. Thus, for Δ_{11} , we first find *G* and G_1 (Fig. 3.44), and then find the set S_1 of all trees of G_1 (Fig. 3.45); then for Δ_{13} , we find G_3 (Fig. 3.46) and the set S_3 of all trees of G_3 (Fig. 3.47) and then from S_1 and S_3 we find all the terms common to both sets (which correspond to the



FIGURE 3.45 (a) The set of all trees of the modified graph G_1 which corresponds to (b) the set of all 2-trees of G (with nodes 1 and R in separate parts in each of such 2-tree).



FIGURE 3.46 The modified graph G_3 of G.

set of all trees common to G_1 and G_3) as shown in Fig. 3.48. Finally we form the ratio of 2-tree admittance products according to Eq. (3.49). Thus from Figs. 3.45 and 3.48, we find

$$\frac{V_0}{V_i} = \frac{Y_a Y_c + Y_a Y_e + Y_b Y_c + Y_c Y_e}{Y_a Y_c + Y_a Y_d + Y_a Y_e + Y_b Y_c + Y_b Y_d + Y_b Y_e + Y_c Y_e + Y_d Y_e}$$

which is identical to the results obtained by the flowgraph techniques.

From the above discussions and Example 3.7 we see that the Mason approach is the best from the systems viewpoint, especially when a single source is involved. It gives an excellent physical insight to the system and reveals the cause-effect relationships at various stages when graph reduction technique is employed. While the Coates approach enables one to compute the output directly regardless of the number of inputs involved in the system, thus overcoming one of the difficulties associated with Mason's approach, it does not allow one to reduce the graph step-by-step toward the final solution as Mason's does. However, it is interesting to note that in the modified Coates technique the introduction of the loop-sets (analogous to trees) and the 2-loop-sets (analogous to 2-trees) brings together the two different concepts—the flowgraph approach and the k-tree approach.

From the networks point of view, the Maxwell *k*-tree approach not only enables one to express the solution in terms of the topology (namely the trees and 2-trees in Example 3.7) of the network but also avoids the cancellation problem inherent in all the flowgraph techniques since, as evident from Example 3.7, the trees and



FIGURE 3.47 (a) The set of all trees of the modified graph G_3 , which corresponds to (b) the set of all 2-trees of G (with nodes 3 and R in separate parts in each of such 2-tree).



FIGURE 3.48 The set of all 2-trees of *G* (with nodes 1 and 3 in one part of the reference node *R* in the other part of each of such 2-tree).

the 2-trees in the gain expression by the *k*-tree approach correspond (one-to-one) to the *uncanceled terms* in the final expressions of the gain by the flowgraph techniques. Finally, it should be obvious that the *k*-tree approach depends upon the knowledge of the graph of a given network. Thus, if in a network problem only the system of (loop or node) equations is given and the network is not known, or more generally, if a system is characterized by a block diagram or a system of equations, the *k*-tree approach cannot be applied and one must resort to the flowgraph techniques between the two approaches.

Some Topological Applications in Network Analysis and Design

In practice a circuit designer often has to make approximations and analyze the same network structure many times with different sets of component values before the final network realization is obtained. Conventional analysis techniques which require the evaluation of high-order determinants are undesirable even on a digital computer because of the large amount of redundancy inherent in the determinant expansion process. The extra calculation in the evaluation (expansion of determinants) and simplification (cancellation of terms) is time consuming and costly and thereby contributes much to the undesirability of such methods.

The *k*-tree topological formulas presented in this section, on the other hand, eliminate completely the cancellation of terms. Also, they are particularly suited for digital computation when the size of the network is not exceedingly large. All of the terms involved in the formulas can be computed by means of a digital compute using a single "tree-finding" program [Chan, 1969]. Thus, the application of topological formulas in analyzing a network with the aid of a digital computer can mean a saving of a considerable amount of time and cost to the circuit designer, especially true when it is necessary to repeat the same analysis procedure a large number of times.

In a preliminary system design, the designer usually seeks one or more concepts which will meet the specifications, and in engineering practice each concept is generally subjected to some form of analysis. For linear systems, the signal flowgraph of Mason is widely used in this activity. The flowgraph analysis is popular because it depicts the relationships existing between system variables, and the graphical structure may be manipulated using Mason's formulas to obtain system transfer functions in symbolic or symbolic/numerical form.

Although the preliminary design problems are usually of limited size (several variables), hand derivation of transfer functions is nonetheless difficult and often prone to error arising from the omission of terms. The recent introduction of remote, time-shared computers into modern design areas offers a means to handle such problems swiftly and effectively.

An efficient algorithm suitable for digital compution of transfer functions from the signal flowgraph description of a linear system has been developed (Dunn and Chan, 1969] which provides a powerful analytical tool in the conceptual phases of linear system design.

In the past several decades, graph theory has been widely used in electrical engineering, computer science, social science, and in the solution of economic problems [Swamy and Thulasiraman, 1981; Chen, 1990]. finally, the application of graph theory in conjunction with symbolic network analysis and computer-aided simulation of electronic circuits has been well recognized in recent years [Lin, 1991].

Defining Terms

Branches of a tree: The edges contained in a tree.

Circuit (or loop): A closed path where all vertices are of degree 2, thus having *no* endpoints in the path. **Circuit-set (or loop-set):** The set of all edges contained in a circuit (loop).

- **Connectedness:** A graph or subgraph is said to be connected if there is at least one path between *every* pair of its vertices.
- Flowgraph G_l (or modified Coates graph G_l): The flowgraph G_l (called *the modified Coates graph*) of a system S of n independent linear equations in n unknowns

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, 2, \dots, n$$

is an oriented graph such that the variable x_j in *S* is represented by a *node* (also denoted by x_j) in G_l , and the coefficient a_{ij} of the variable x_j in *S* by a *branch* with a branch gain a_{ij} connected between nodes x_i and x_j in G_l and directed from x_j to x_i . Furthermore, a *source node* l is included in G_l such that for each constant b_k in *S* there is a node with gain b_k in G_l from node l to node s_k . Graph G_{ij} is the subgraph of G_l obtained by first removing all the outgoing branches from node x_j and then short-circuiting node l to node x_j . A *loop set* l is a subgraph of G_{l0} that contains all the nodes of G_{l0} with each node having exactly one incoming and one outgoing branch. The product p of the gains of all the branches in l is called a *loop-set product*. A 2-*loop-set* l_2 is a subgraph of G_{lj} containing all the nodes of G_{lj} with each node having exactly one incoming and one outgoing branch. The product p_2 of the gains of all the branches in l_2 is called a 2-*loop-set product*.

k-tree admittance product of a *k*-tree: The product of the admittances of all the branches of the *k*-tree.

- *k*-tree of a connected graph *G*: A proper subgraph of *G* consisting of *k* unconnected circuitless subgraphs, each subgraph itself being connected, which together contain all the vertices of *G*.
- **Linear graph:** A set of line segments called edges and points called vertices, which are the endpoints of the edges, interconnected in such a way that the edges are connected to (or incident with) the vertices. The degree of a vertex of a graph is the number of edges incident with that vertex.
- **Path:** A subgraph having all vertices of degree 2 except for the two endpoints which are of degree 1 and are called the terminals of the path, where the degree of a vertex is the number of edges connected to the vertex in the subgraph.

Path-set: The set of all edges in a path.

Proper subgraph: A subgraph which does not contain all of the edges of the given graph.

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Signal-flowgraph G_m (or Mason's graph G_m): A signal-flowgraph G_m of a system S of n independent linear (algebraic) equations in n unknowns

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, 2, \dots, n$$

is a graph with junction points called *nodes* which are connected by directed line segments called *branches* with signals traveling along the branches only in the direction described by the arrows of the branches. A signal x_k traveling along a branch between x_k and x_j is multiplied by the gain of the branches g_{kj} , so that a signal $g_{kj} x_k$ is delivered at node x_j . An *input node* (*source*) is a node which contains only outgoing branches; an *output node* (*sink*) is a node which has only incoming branches. A *path* is a continuous unidirectional succession of branches, all of which are traveling in the same direction; a *forward path* is a path from the input node to the output node along which all nodes are encountered exactly once; and a *feedback path* (*loop*) is a closed path which originates from and terminates at the same node, and along with all other nodes are encountered exactly once (the trivial case is a *self-loop* which contains exactly once and any branches in a branches in a branches in a long.

one node and one branch). A *path* gain is the product of all the branch gains of the branches in a loop. **Subgraph:** A subset of the edges of a given graph.

Tree: A connected subgraph of a given connected graph G which contains all the vertices of G but no circuits.

Related Topic

3.2 Node and Mesh Analysis

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Further Information

All defining terms used in this section can be found in S.P. Chan, *Introductory Topological Analysis of Electrical Networks*, Holt, Rinehart and Winston, New York, 1969. Also an excellent reference for the applications of graph

theory in electrical engineering (i.e., network analysis and design) is S. Seshu and M.B. Reed, *Linear Graphs and Electrical Networks*, Addison-Wesley, Reading, Mass., 1961.

For applications of graph theory in computer science, see M.N.S. Swamy and K. Thulasiraman, *Graphs, Networks, and Algorithms*, Wiley, New York, 1981.

For flowgraph applications, see W.K. Chen, *Theory of Nets: Flows in Networks*, Wiley Interscience, New York, 1990.

For applications of graph theory in symbolic network analysis, see P.M. Lin, *Symbolic Network Analysis*, Elsevier, New York, 1991.

3.7 **Two-Port Parameters and Transformations**

Norman S. Nise

Introduction

Many times we want to model the behavior of an electric network at only two terminals as shown in Fig. 3.49. Here, only V_1 and I_1 , not voltages and currents internal to the circuit, need to be described. To produce the model for a linear circuit, we use **Thévenin's** or **Norton's theorem** to simplify the network as viewed from the selected terminals. We define the pair of terminals shown in Fig. 3.49 as a **port**, where the current, I_1 , entering one terminal equals the current leaving the other terminal.

If we further restrict the network by stating that (1) all external connections to the circuit, such as sources and impedances, are made at the port and (2) the network can have internal **dependent sources**, but not **independent sources**, we can mathematically model the network at the port as



FIGURE 3.49 An electrical net-work port.

$$V_1 = ZI_1 \tag{3.50}$$

or

$$I_1 = YV_1 \tag{3.51}$$

where Z is the Thévenin impedance and Y is the Norton admittance at the terminals. Z and Y can be constant resistive terms, Laplace transforms Z(s) or Y(s), or sinusoidal steady-state functions $Z(j\omega)$ or $Y(j\omega)$.

Defining Two-Port Networks

Electrical networks can also be used to transfer signals from one port to another. Under this requirement, connections to the network are made in two places, the input and the output. For example, a transistor has an input between the base and emitter and an output between the collector and emitter. We can model such circuits as **two-port networks** as shown in Fig. 3.50. Here we see the input port, represented by V_1 and I_1 , and the output port, represented by V_2 and I_2 . Currents are assumed positive if they flow as shown in Fig. 3.50. The



FIGURE 3.50 A two-port network.

same restrictions about external connections and internal sources mentioned above for the single port also apply. Now that we have defined two-port networks, let us discuss how to create a mathematical model of the network by establishing relationships among all of the input and output voltages and currents. Many possibilities exist for modeling. In the next section we arbitrarily begin by introducing the *z*-parameter model to establish the technique. In subsequent sections we present alternative models and draw relationships among them.

Mathematical Modeling of Two-Port Networks via z Parameters

In order to produce a mathematical model of circuits represented by Fig. 3.50, we must find relationships among V_1 , I_1 , V_2 , and I_2 . Let us visualize placing a current source at the input and a current source at the output. Thus, we have selected two of the variables, I_1 and I_2 . We call these variables the independent variables. The remaining variables, V_1 and V_2 , are dependent upon the selected applied currents. We call V_1 and V_2 the dependent variables. Using **superposition** we can write each dependent variable as a function of the independent variables as follows:

$$V_1 = z_{11}I_1 + z_{12}I_2 \tag{3.52a}$$

$$V_2 = z_{21}I_1 + z_{22}I_2 \tag{3.52b}$$

We call the coefficients, z_{ij} , in Eqs. (3.52) parameters of the two-port network or, simply, two-port parameters.

From Eqs. (3.52), the two-port parameters are evaluated as

$$z_{11} = \frac{V_1}{I_1}\Big|_{I_2=0}; \quad z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0}$$

$$z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0}; \quad z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0}$$
(3.53)

Notice that each parameter can be measured by setting a port current, I_1 or I_2 , equal to zero. Since the parameters are found by setting these currents equal to zero, this set of parameters is called **open-circuit parameters**. Also, since the definitions of the parameters as shown in Eqs. (3.53) are the ratio of voltages to currents, we alternatively refer to them as **impedance parameters**, or *z* **parameters**. The parameters themselves can be impedances represented as Laplace transforms, Z(s), sinusoidal steady-state impedance functions, $Z(j\omega)$, or simply pure resistance values, *R*.

Evaluating Two-Port Network Characteristics in Terms of z Parameters

The two-port parameter model can be used to find the following characteristics of a two-port network when used in some cases with a source and load as shown in Fig. 3.51:

Input impedance =
$$Z_{in} = V_1 / I_1$$
 (3.54a)

Output impedance =
$$Z_{out} = V_2/I_2 | V_s = 0$$
 (3.54b)

Network voltage gain =
$$V_g = V_2/V_1$$
 (3.54c)

Total voltage gain =
$$V_{gt} = V_2 / V_S$$
 (3.54d)

Network current gain =
$$I_g = I_2/I_1$$
 (3.54e)

To find Z_{in} of Fig. 3.51, determine V_1/I_1 . From Fig. 3.51, $V_2 = -I_2 Z_L$. Substituting this value in Eq. 3.52(b) and simplifying, Eqs. (3.52) become

$$V_1 = z_{11}I_1 + z_{12}I_2 \tag{3.55a}$$

$$0 = z_{21}I_1 + (z_{22} + Z_L)I_2$$
 (3.55b)

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FIGURE 3.51 Terminated two-port network for finding two-port network characteristics.

Solving simultaneously for I_1 and then forming $V_1/I_1 = Z_{in}$, we obtain

$$Z_{\rm in} = \frac{V_1}{I_1} = z_{11} - \frac{z_{12}z_{21}}{\left(z_{22} + Z_L\right)}$$
(3.56)

To find Z_{out} , set $V_s = 0$ in Fig. 3.51. This step terminates the input with Z_s . Next, determine V_2/I_2 . From Fig. 3.51 with V_s shorted, $V_1 = -I_1Z_s$. By substituting this value into Eq. 3.52(a) and simplifying, Eqs. (3.52) become

$$0 = (z_{11} + Z_s)I_1 + z_{12}I_2$$
(3.57a)

$$V_2 = z_{21}I_1 + z_{22}I_2 \tag{3.57b}$$

By solving simultaneously for I_2 and then forming $V_2/I_2 = Z_{out}$,

$$Z_{\text{out}} = \frac{V_2}{I_2}\Big|_{V_s=0} = z_{22} - \frac{z_{12}z_{21}}{(z_{11} + Z_s)}$$
(3.58)

To find V_g , we see from Fig. 3.51 that $I_2 = -V_2/Z_L$. Substituting this value in Eqs. (3.52) and simplifying, we obtain

$$V_1 = z_{11}I_1 - \frac{z_{12}}{Z_L}V_2$$
(3.59a)

$$0 = z_{21}I_1 - \left(\frac{z_{22} + Z_L}{Z_L}\right)V_2$$
(3.59b)

By solving simultaneously for V_2 and then forming $V_2/V_1 = V_g$,

$$V_{g} = \frac{V_{2}}{V_{1}} = \frac{z_{21}Z_{L}}{z_{11}(z_{22} + Z_{L}) - z_{12}z_{21}}$$
(3.60)

Similarly, other characteristics, such as current gain and the total voltage gain from the source voltage to the load voltage can be found. Table 3.1 summarizes many of the network characteristics that can be found using z parameters as well as the process to arrive at the result.

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TABLE 3.1	Network	Characteristics	Developed	from	z-Parameter	Defining	Eqs.	(3.52)	
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Network Characteristic Definition	From Fig. 3.51	Substitute in Defining Eqs. (3.52) and Obtain	Solve for Network Characteristic
Input impedance $z_{\rm in} = \frac{V_1}{I_1}$	$V_2 = -I_2 Z_L$	$V_{1} = z_{11}I_{1} + z_{12}I_{2}$ $0 = z_{21}I_{1} + (z_{22} + Z_{L})I_{2}$	$Z_{\rm in} = z_{11} - \frac{z_{12}z_{21}}{z_{22} + Z_L}$
Output impedance $Z_{\text{out}} = \frac{V_2}{I_2} \Big _{V_s=0}$	$V_1 = V_S - I_1 Z_S$ $V_S = 0$	$0 = (z_{11} + Z_s) I_1 + z_{12} I_2$ $V_2 = z_{21} I_1 + z_{22} I_2$	$Z_{\text{out}} = z_{22} - \frac{z_{12}z_{21}}{z_{11} + Z_s}$
Network voltage gain $V_g = \frac{V_2}{V_1}$	$I_2 = -\frac{V_2}{Z_L}$	$V_{1} = z_{11}I_{1} - \frac{z_{12}V_{2}}{Z_{L}}$ $0 = z_{21}I_{1} - \frac{(z_{22} + Z_{L})V_{2}}{Z_{L}}$	$V_g = \frac{z_{21} Z_L}{z_{11} (z_{22} + Z_L) - z_{12} z_{21}}$
Total voltage gain $V_{\rm gt} = \frac{V_2}{V_S}$	$V_1 = V_s - I_1 Z_s$ $I_2 = -\frac{V_2}{Z_L}$	$V_{S} = (z_{11} + Z_{S}) I_{1} - \frac{z_{12}V_{2}}{Z_{L}}$ $0 = z_{21}I_{1} - \frac{(z_{22} + Z_{L})V_{2}}{Z_{L}}$	$V_{\rm gt} = \frac{z_{21}Z_L}{(z_{11} + Z_S)(z_{22} + Z_L) - z_{12}z_{21}}$
Network current gain $I_g = \frac{I_2}{I_1}$	$V_2 = -I_2 Z_L$	$V_{1} = z_{11}I_{1} + z_{12}I_{2}$ $0 = z_{21}I_{1} + (z_{22} + Z_{L})I_{2}$	$I_g = -\frac{z_{21}}{z_{22} + Z_L}$

To summarize the process of finding network characteristics:

- 1. Define the network characteristic.
- 2. Use approproate relationships from Fig. 3.51.
- 3. Substitute the relationships from Step 2 into Eqs. (3.52).
- 4. Solve the modified equations for the network characteristic.

An Example Finding z Parameters and Network Characteristics

To solve for two-port network characteristics we can first represent the network with its two-port parameters and then use these parameters to find the characteristics summarized in Table 3.1. To find the parameters, we terminate the network adhering to the definition of the parameter we are evaluating. Then, we can use mesh or nodal analysis, current or voltage division, or equivalent impedance to solve for the parameters. The following example demonstates the technique.

Consider the network of Fig. 3.52(a). The first step is to evaluate the *z* parameters. From their definition, z_{11} and z_{21} are found by open-circuiting the output and applying a voltage at the input as shown in Fig. 3.52(b). Thus, with $I_2 = 0$

$$6I_1 - 4I_a = V_1 \tag{3.61a}$$

$$-4I_1 + 18I_a = 0 (3.61b)$$



FIGURE 3.52 (a) Two-port network example; (b) two-port network modified to find z_{11} and z_{21} ; (c) two-port network modified to find z_{22} and z_{12} .

Solving for I_1 yields

$$I_{1} = \begin{vmatrix} V_{1} & -4 \\ 0 & 18 \\ \hline 6 & -4 \\ -4 & 18 \end{vmatrix} = \frac{18V_{1}}{92}$$
(3.62)

from which

$$z_{11} = \frac{V_1}{I_1} \bigg|_{I_2=0} = \frac{46}{9}$$
(3.63)

We now find z_{21} . From Eq. (3.61b)

$$\frac{I_a}{I_1} = \frac{2}{9}$$
(3.64)

But, from Fig. 3.52(b), $I_a = V_2/8$. Thus,

$$z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0} = \frac{16}{9}$$
(3.65)

Based on their definitions, z_{22} and z_{12} are found by placing a source at the output and open-circuiting the input as shown in Fig. 3.52(c). The equivalent resistance, R_{2eq} , as seen at the output with $I_1 = 0$ is

$$R_{2eq} = \frac{8 \times 10}{8 + 10} = \frac{40}{9}$$
(3.66)

Therefore,

$$z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0} = \frac{40}{9}$$
(3.67)

From Fig. 3.52(c), using voltage division

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$$V_1 = (4/10) V_2 \tag{3.68}$$

But

$$V_2 = I_2 R_{2eq} = I_2(40/9)$$
(3.69)

Substituting Eq. (3.69) into Eq. (3.68) and simplifying yields

$$z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0} = \frac{16}{9}$$
(3.70)

Using the z-parameter values found in Eqs. (3.63), (3.65), (3.67), and (3.70) and substituting into the network characteristic relationships shown in the last column of Table 3.1, assuming $Z_s = 20 \ \Omega$ and $Z_L = 10 \ \Omega$, we obtain $Z_{in} = 4.89 \ \Omega$, $Z_{out} = 4.32 \ \Omega$, $V_g = 0.252$, $V_{gt} = 0.0494$, and $I_g = -0.123$.

Additional Two-Port Parameters and Conversions

We defined the *z* parameters by establishing I_1 and I_2 as the independent variables and V_1 and V_2 as the dependent variables. Other choices of independent and dependent variables lead to definitions of alternative two-port parameters. The total number of combinations one can make with the four variables, taking two at a time as independent variables, is six. Table 3.2 defines the six possibilities as well as the names and symbols given to the parameters.

The table also presents the expressions used to calculate directly the parameters of each set based upon their definition as we did with *z* parameters. For example, consider the *y*, or **admittance**, **parameters**. These parameters are seen to be **short-circuit parameters**, since their evaluation requires V_1 or V_2 to be zero. Thus, to find y_{22} we short-circuit the input and find the admittance looking back from the output. For Fig. 3.52(a), $y_{22} = 23/88$. Any parameter in Table 3.2 is found either by open-circuiting or short-circuiting a terminal and then performing circuit analysis to find the defining ratio.

Another method of finding the parameters is to convert from one set to another. Using the "Definition" row in Table 3.2, we can convert the defining equations of one set to the defining equations of another set. For example, we have already found the z parameters. We can find the h parameters as follows:

Solve for I_2 using the second *z*-parameter equation, Eq. (3.52b), and obtain the second *h*-parameter equation as

$$I_2 = -\frac{z_{21}}{z_{22}} I_1 + \frac{1}{z_{22}} V_2$$
(3.71)

which is of the form, $I_2 = h_{21}I_1 + h_{22}V_2$, the second *h*-parameter equation. Now, substitute Eq. (3.71) into the first *z*-parameter equation, Eq. (3.52a), rearrange, and obtain

$$V_1 = \frac{z_{11}z_{22} - z_{12}z_{21}}{z_{22}} I_1 + \frac{z_{12}}{z_{22}} V_2$$
(3.72)

which is of the form, $V_1 = h_{11}I_1 + h_{12}V_2$, the first *h*-parameter equation. Thus, for example, $h_{21} = -z_{21}/z_{22}$ from Eq. (3.71). Other transformations are found through similar manipulations and are summarized in Table 3.2.

Finally, there are other parameter sets that are defined differently from the standard sets covered here. Specifically, they are scattering parameters used for microwave networks and image parameters used for filter design. A detailed discussion of these parameters is beyond the scope of this section. The interested reader should consult the bibliography in the "Further Information" section below, or Section 39.1 of this handbook.

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	Impedance Parameters (Open-Circuit Parameters) <i>z</i>	Admittance Parameters (Short-Circuit Parameters) <i>y</i>	Hybrid Parameters h
Definition	$V_{1} = z_{11}I_{1} + z_{12}I_{2}$ $V_{2} = z_{21}I_{1} + z_{22}I_{2}$	$I_1 = y_{11}V_1 + y_{12}V_2$ $I_2 = y_{21}V_1 + y_{22}V_2$	$V_1 = h_{11}I_1 + h_{12}V_2$ $I_2 = h_{21}I_1 + h_{22}V_2$
Parameters	$z_{11} = \frac{V_1}{I_1}\Big _{I_2=0}; \ z_{12} = \frac{V_1}{I_2}\Big _{I_1=0}$	$y_{11} = \frac{I_1}{V_1}\Big _{V_2=0}; \ y_{12} = \frac{I_1}{V_1}\Big _{V_1=0}$	$h_{11} = \frac{V_1}{I_1}\Big _{V_2=0}; \ h_{12} = \frac{V_1}{V_2}\Big _{I_1=0}$
	$z_{21} = \frac{V_2}{I_1}\Big _{I_2=0}; \ z_{22} = \frac{V_2}{I_2}\Big _{I_1=0}$	$y_{21} = \frac{I_2}{V_1} \bigg _{V_2=0}; y_{22} = \frac{I_2}{V_1} \bigg _{V_1=0}$	$h_{21} = \frac{I_2}{I_1}\Big _{V_2=0}; h_{22} = \frac{I_2}{V_2}\Big _{I_1=0}$
Conversion to <i>z</i> parameters		$z_{11} = \frac{y_{22}}{\Delta_y}; z_{12} = \frac{-y_{12}}{\Delta_y}$	$z_{11} = \frac{\Delta_h}{h_{22}}; \qquad z_{12} = \frac{h_{12}}{h_{22}}$
		$z_{21} = \frac{-y_{21}}{\Delta_y}; \ z_{22} = \frac{y_{11}}{\Delta_y}$	$z_{21} = \frac{-h_{21}}{h_{22}}; z_{22} = \frac{1}{h_{22}}$
Conversion to <i>y</i> parameters	$y_{11} = \frac{z_{22}}{\Delta_z}; \qquad y_{12} = \frac{-z_{12}}{\Delta_z}$		$y_{11} = \frac{1}{h_{11}}; y_{12} = \frac{-h_{12}}{h_{11}}$
	$y_{21} = \frac{-z_{21}}{\Delta_z}; y_{22} = \frac{z_{11}}{\Delta_z}$		$y_{21} = \frac{h_{21}}{h_{11}}; y_{22} = \frac{\Delta_h}{h_{11}}$
Conversion to <i>h</i> parameters	$h_{11} = \frac{\Delta_z}{z_{22}}; h_{12} = \frac{z_{12}}{z_{22}}$	$h_{11} = \frac{1}{y_{11}}; h_{12} = \frac{-y_{12}}{y_{11}}$	
	$h_{21} = \frac{-z_{21}}{z_{22}}; h_{22} = \frac{1}{z_{22}}$	$h_{21} = \frac{y_{21}}{y_{11}}, h_{22} = \frac{\Delta_y}{y_{11}}$	
Conversion to <i>g</i> parameters	$g_{11} = \frac{1}{z_{11}}; g_{12} = \frac{-z_{12}}{z_{11}}$	$g_{11} = \frac{\Delta_y}{y_{22}}; g_{12} = \frac{y_{12}}{y_{22}}$	$g_{11} = \frac{h_{22}}{\Delta_h}; g_{12} = \frac{-h_{12}}{\Delta_h}$
	$g_{21} = \frac{z_{21}}{z_{11}}; g_{22} = \frac{\Delta_z}{z_{11}}$	$g_{21} = \frac{-y_{21}}{y_{22}}; g_{22} = \frac{1}{y_{22}}$	$g_{21} = \frac{-h_{21}}{\Delta_h}; g_{22} = \frac{h_{11}}{\Delta_h}$
Conversion to <i>T</i> parameters	$A = \frac{z_{11}}{z_{21}}; B = \frac{\Delta_z}{z_{21}}$	$A = \frac{-y_{22}}{y_{21}}; B = \frac{-1}{y_{21}}$	$A = \frac{-\Delta_h}{h_{21}}; B = \frac{-h_{11}}{h_{21}}$
	$C = \frac{1}{z_{21}}; D = \frac{z_{22}}{z_{21}}$	$C = \frac{-\Delta_y}{y_{21}}; D = \frac{-y_{11}}{y_{21}}$	$C = \frac{-h_{22}}{h_{21}}; D = \frac{-1}{h_{21}}$
Conversion to <i>T'</i> parameters	$A' = \frac{z_{22}}{z_{12}}; B' = \frac{\Delta_z}{z_{12}}$	$A' = \frac{-y_{11}}{y_{12}}; \ B' = \frac{-1}{y_{12}}$	$A' = \frac{1}{h_{12}}; B' = \frac{h_{11}}{h_{12}}$
	$C' = \frac{1}{z_{12}}; D' = \frac{z_{11}}{z_{12}}$	$C' = \frac{-\Delta_y}{y_{12}}; D' = \frac{-y_{22}}{y_{12}}$	$C' = \frac{h_{22}}{h_{12}}; D' = \frac{\Delta_h}{h_{12}}$
Δ	$\Delta_z = z_{11} z_{22} - z_{12} z_{21}$	$\Delta_{y} = y_{11}y_{22} - y_{12}y_{21}$	$\Delta_h = h_{11}h_{22} - h_{12}h_{21}$

TABLE 3.2 Two-Port Parameter Definitions and Conversions

	Inv. hybrid parameters g	Transmission parameters T	Inv. transmission par. <i>T</i> '
Definition	$I_{1} = g_{11}V_{1} + g_{12}I_{2}$ $V_{1} = g_{11}V_{1} + g_{12}I_{2}$	$V_1 = AV_2 - BI_2$ $I_2 = CV_2 - DI_2$	$V_2 = A'V_1 - B'I_1$ $I = C'V - D'I$
Parameters	$\begin{aligned} \mathbf{y}_{2} &= \left. g_{21} \mathbf{y}_{1} + \left. g_{22} \mathbf{z}_{2} \right. \\ g_{11} &= \left. \frac{I_{1}}{V_{1}} \right _{I_{2}=0}; g_{12} &= \left. \frac{I_{1}}{I_{2}} \right _{V_{1}=0} \end{aligned}$	$A_{1} = \frac{V_{1}}{V_{2}}\Big _{I_{2}=0}; B = \frac{-V_{1}}{I_{2}}\Big _{V_{2}=0}$	$A' = \frac{V_2}{V_1} \bigg _{I_1=0}; B' = \frac{-V_2}{I_1} \bigg _{V_1=0}$
	$g_{21} = \frac{V_2}{V_1}\Big _{I_2=0}; g_{22} = \frac{V_2}{I_2}\Big _{V_1=0}$	$C = \frac{I_1}{V_2}\Big _{I_2=0}; D = \frac{-I_1}{I_2}\Big _{V_2=0}$	$C' = \frac{I_2}{V_1}\Big _{I_1=0}; D' = \frac{-I_2}{I_1}\Big _{V_1=0}$
Conversion to <i>z</i> parameters	$z_{11} = \frac{1}{g_{11}}; z_{12} = \frac{-g_{12}}{g_{11}}$	$z_{11} = \frac{A}{C}; z_{12} = \frac{\Delta_T}{C}$	$z_{11} = \frac{D'}{C'}; z_{12} = \frac{1}{C'}$
	$z_{21} = \frac{g_{21}}{g_{11}}; z_{22} = \frac{\Delta_g}{g_{11}}$	$z_{21} = \frac{1}{C}; z_{22} = \frac{D}{C}$	$z_{21} = \frac{\Delta_{T'}}{C'}; \ z_{22} = \frac{A'}{C'}$
Conversion to <i>y</i> parameters	$y_{11} = \frac{\Delta_g}{g_{22}}; y_{12} = \frac{g_{12}}{g_{22}}$	$y_{11} = \frac{D}{B}; y_{12} = \frac{-\Delta_T}{B}$	$y_{11} = \frac{A'}{B'}; y_{12} = \frac{-1}{B'}$
	$y_{21} = \frac{-g_{21}}{g_{22}}; y_{22} = \frac{1}{g_{22}}$	$y_{21} = \frac{-1}{B}; y_{22} = \frac{A}{B}$	$y_{21} = \frac{-\Delta_{T'}}{B'}; y_{22} = \frac{D'}{B'}$
Conversion to <i>h</i> parameters	$h_{11} = \frac{g_{22}}{\Delta_g}; h_{12} = \frac{-g_{12}}{\Delta_g}$	$h_{11} = \frac{B}{D}; h_{12} = \frac{\Delta_T}{D}$	$h_{11} = \frac{B'}{A'}; h_{12} = \frac{1}{A'}$
	$h_{21} = \frac{-g_{21}}{\Delta_g}; \ h_{22} = \frac{g_{11}}{\Delta_g}$	$h_{21} = \frac{-1}{D}; \ h_{22} = \frac{C}{D}$	$h_{21} = \frac{-\Delta_{T'}}{A'}; h_{22} = \frac{C'}{A'}$
Conversion to <i>g</i> parameters		$g_{11} = \frac{C}{A}; g_{12} = \frac{-\Delta_T}{A}$	$g_{11} = \frac{C'}{D'}; g_{12} = \frac{-1}{D'}$
		$g_{21} = \frac{1}{A}; g_{22} = \frac{B}{A}$	$g_{21} = \frac{\Delta_{T'}}{D'}; g_{22} = \frac{B'}{D'}$
Conversion to <i>T</i> parameters	$A = \frac{1}{g_{21}}; \ B = \frac{g_{22}}{g_{21}}$		$A = \frac{D'}{\Delta_{T'}}; B = \frac{B'}{\Delta_{T'}}$
	$C = \frac{g_{11}}{g_{21}}; D = \frac{\Delta_g}{g_{21}}$		$C = \frac{C'}{\Delta_{T'}}; D = \frac{A'}{\Delta_{T'}}$
Conversion to <i>T'</i> parameters	$A' = \frac{-\Delta_g}{g_{12}}; \ B' = \frac{-g_{22}}{g_{12}}$	$A' = \frac{D}{\Delta_T}; \ B' = \frac{B}{\Delta_T}$	
	$C' = \frac{-g_{11}}{g_{12}}; D' = \frac{-1}{g_{12}}$	$C' = \frac{C}{\Delta_T}; \ D' = \frac{A}{\Delta_T}$	
Δ	$\Delta_g = g_{11}g_{22} - g_{12}g_{21}$	$\Delta_T = AD - BC$	$\Delta_{T'} = A'D' - B'C'$

TABLE 3.2 (continued)	Two-Port Parameter Definitions and Conversions
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Adapted from Van Valkenburg, M.E. 1974. *Network Analysis*, 3rd ed. Table 11-2, p. 337. Prentice-Hall, Englewood Cliffs, NJ. With permission.

Two-Port Parameter Selection

The choice of parameters to use for a particular analysis or design problem is based on analytical convenience or the physics of the device or network at hand. For example, an ideal transformer cannot be represented with z parameters. I_1 and I_2 are not linearly independent variables, since they are related through the turns ratio. A similar argument applies to the *y*-parameter representation of a transformer. Here V_1 and V_2 are not independent, since they too are related via the turns ratio. A possible choice for the transformer is the **transmission parameters**. For an ideal transformer, B and C would be zero. For a BJT transistor, there is effectively linear independence between the input current and the output voltage. Thus, the hybrid parameters are the parameters of choice for the transistor.

The choice of parameters can be based also upon the ease of analysis. For example, Table 3.3 shows that "T" networks lend themselves to easy evaluation of the *z* parameters, while *y* parameters can be easily evaluated for " Π " networks. Table 3.3 summarizes other suggested uses and selections of network parameters for a few specific cases. When electric circuits are interconnected, a judicious choice of parameters can simplify the calculations to find the overall parameter description for the interconnected networks. For example, Table 3.3 shows that the *z* parameters for series-connected networks are simply the sum of the *z* parameters of the individual circuits (see Ruston et al., [1966] for derivations of the parameters for some of the interconnected networks). The bold entries imply 2×2 matrices containing the four parameters. For example,

$$\mathbf{h} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$
(3.73)

Summary

In this section, we developed two-port parameter models for two-port electrical networks. The models define interrelationships among the input and output voltages and currents. A total of six models exists, depending upon which two variables are selected as independent variables. Any model can be used to find such network characteristics as input and output impedance, and voltage and current gains. Once one model is found, other models can be obtained from transformation equations. The choice of parameter set is based upon physical reality and analytical convenience.

Defining Terms

- Admittance parameters: That set of two-port parameters, such as *y* parameters, where all the parameters are defined to be the ratio of current to voltage. See Table 3.2 for the specific definition.
- **Dependent source:** A voltage or current source whose value is related to another voltage or current in the network.
- g Parameters: See hybrid parameters.
- *h* Parameters: See hybrid parameters.
- **Hybrid (inverse hybrid) parameters:** That set of two-port parameters, such as h(g) parameters, where input current (voltage) and output voltage (current) are the independent variables. The parenthetical expressions refer to the inverse hybrid parameters. See Table 3.2 for specific definitions.
- **Impedance parameters:** That set of two-port parameters, such as *z* parameters, where all the parameters are defined to be the ratio of voltage to current. See Table 3.2 for the specific definition.
- **Independent source:** A voltage or current source whose value is not related to any other voltage or current in the network.
- **Norton's theorem:** At a pair of terminals a linear electrical network can be replaced with a current source in parallel with an admittance. The current source is equal to the current that flows through the terminals when the terminals are short-circuited. The admittance is equal to the admittance at the terminals with all independent sources set equal to zero.
- **Open-circuit parameters:** Two-port parameters, such as *z* parameters, evaluated by open-circuiting a port.

	Common Circuit Applications	Interconnected Network Applications
Impedance parameters z	• T networks	• Series connected
	Z_{a} Z_{b} I_{1} Z_{c} V_{1} Z_{c} V_{2} I_{1} $Z_{11} = Z_{a} + Z_{b}$ $Z_{12} = Z_{21} = Z_{c}$ $Z_{22} = Z_{b} + Z_{c}$	$z_{B} = z_{A} + z_{B}$
Admittance parameters y	• П networks	Parallel connected
	$\begin{array}{c} Y_{c} \\ \downarrow \\ V_{1} \\ I_{1} \\ \downarrow \\ Y_{a} \\ Y_{b} \\ \downarrow \\ \downarrow \\ I_{2} \\ \downarrow \\ I_{2} \\ I_{$	\mathbf{y}_{A} \mathbf{y}_{A} \mathbf{y}_{A} \mathbf{y}_{A} \mathbf{y}_{A} \mathbf{y}_{B} \mathbf{y}_{A} \mathbf{y}_{B}
	• Field effect transistor equivalent circuit • I_1 $V_1 = 1/y_{11}$ $V_2 = V_2$ $V_2 = V_2$ $V_1 = 1/y_{22}$ $V_2 = V_2$ $V_2 = V_2$ $V_1 = 1/y_{22}$ $V_2 = V_2$ $V_2 = V_2$ $V_2 = V_2$ $V_2 = V_2$ $V_2 = V_2$ $V_2 = V_2$	
	$1/y_{11} = \infty, y_{12} = 0, y_{21} = g_m, 1/y_{22} = r_d$	
Hybrid parameters <i>h</i>	• Transistor equivalent circuit	• Series–parallel connected
	$ \begin{array}{c} \begin{array}{c} h_{11} \\ \hline \\ \mu_{1} \\ \hline \\ \mu_{1} \\ \hline \\ \mu_{1} \\ \hline \\ \mu_{2} \\ \hline \\ \mu_{2$	$\mathbf{h}_{\mathbf{h}}_{\mathbf{h}_{\mathbf{h}_{\mathbf{h}_{\mathbf{h}_{\mathbf{h}}}}}}}}}}$
Inverse hybrid parameters g		• Parallel-series connected
		$+ I_1 \qquad g_A \qquad I_2 \qquad + V_2 \qquad I_1 \qquad g_B \qquad I_2$
		$\mathbf{g} = \mathbf{g}_{\mathrm{A}} + \mathbf{g}_{\mathrm{B}}$
Transmission parameters T	• Ideal transformer circuits	• Cascade connected $V_1 I_1 = I_1 I_2 V_2$
		$T = T_A T_B$
Inverse transmission parameters T'		• Cascade connected
		$\stackrel{* \longrightarrow I_1}{\underset{V_1 I_1 \longrightarrow }{\longrightarrow}} \mathbf{T}_{\mathbf{A}}^* \mathbf{T}_{\mathbf{B}}^* \underbrace{I_2 \longrightarrow I_2 V_2}_{I_2 V_2}$
		$\mathbf{T'} = \mathbf{T'_B} \; \mathbf{T'_A}$

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- **Port:** Two terminals of a network where the current entering one terminal equals the current leaving the other terminal.
- **Short-circuit parameters:** Two-port parameters, such as *y* parameters, evaluated by short-circuiting a port.
- **Superposition:** In linear networks, a method of calculating the value of a dependent variable. First, the value of the dependent variable produced by each independent variable acting alone is calculated. Then, these values are summed to obtain the total value of the dependent variable.
- **Thévenin's theorem:** At a pair of terminals a linear electrical network can be replaced with a voltage source in series with an impedance. The voltage source is equal to the voltage at the terminals when the terminals are open-circuited. The impedance is equal to the impedance at the terminals with all independent sources set equal to zero.

T parameters: See transmission parameters.

T' parameters: See transmission parameters.

Transmission (inverse transmission) parameters: That set of two-port parameters, such as the T(T) parameters, where the dependent variables are the input (output) variables of the network and the independent variables are the output (input) variables. The parenthetical expressions refer to the inverse transmission parameters. See Table 3.2 for specific definitions.

Two-port networks: Networks that are modeled by specifying two ports, typically input and output ports.

- **Two-port parameters:** A set of four constants, Laplace transforms, or sinusoidal steady-state functions used in the equations that describe a linear two-port network. Some examples are *z*, *y*, *h*, *g*, *T*, and *T'* parameters.
- *y* **Parameters:** See admittance parameters.

z Parameters: See impedance parameters.

Related Topic

3.3 Network Theorems

References

- H. Ruston and J. Bordogna, "Two-port networks," in *Electric Networks: Functions, Filters, Analysis*, New York: McGraw-Hill, 1966, chap. 4, pp. 244–266.
- M. E. Van Valkenburg, "Two-port parameters," in *Network Analysis*, 3rd ed. Chap. 11, Englewood Cliffs, N.J.: Prentice-Hall, 1974, pp. 325–350.

Further Information

The following texts cover standard two-port parameters:

- J. W. Nilsson, "Two-port circuits," in *Electric Circuits*, 4th ed., Reading Mass.: Addison-Wesley, 1995, chap. 21, pp. 755–786.
- H. Ruston and J. Bordogna, "Two-port networks," in *Electric Networks: Functions, Filters, Analysis*, New York: McGraw-Hill, 1966, chap. 4, pp. 206–311.

The following texts have added coverage of scattering and image parameters:

- H. Ruston and J. Bordogna, "Two-port networks," in *Electric networks: Functions, Filters, Analysis*, New York: McGraw-Hill, 1966, chap. 4, pp. 266–297.
- S. Seshu and N. Balabanian, "Two-port networks," and "Image parameters and filter theory," in *Linear Network Analysis*, New York: Wiley, 1959, chaps. 8 and 11, pp. 291–342, 453–504.

The following texts show applications to electronic circuits:

- F. H. Mitchell, Jr. and F. H. Mitchell, Sr., "Midrange AC amplifier design," in *Introduction to Electronics Design*, Englewood Cliffs, N.J.: Prentice-Hall, 1992, chap. 7, pp. 335–384.
- C. J. Savant, Jr., M. S. Roden, and G. L. Carpenter, "Bipolar transistors," "Design of bipolar junction transistor amplifiers," and "Field-effect transistor amplifiers," in *Electronic Design*, 2nd ed., Redwood City, Calif.: Benjamin/Cummings, 1991, chaps. 2, 3, and 4, pp. 69–212.
- S. S. Sedra and K. C. Smith, "Frequency response" and "Feedback," in *Microelectronic Circuits*, 3rd ed., Philadelphia, Pa.: Saunders, 1991, chaps. 7 and 8, pp. 488–645.