# 1

# *Introduction*

Research in *nonlinear systems and complexity* had made remarkable progress in the 1970's and 1980's, leading to discoveries which were not only new, but also revolutionary in the sense that some of our traditional beliefs regarding the behavior of deterministic systems were relentlessly challenged [63, 64, 79, 92]. Most striking of all, simple deterministic systems can behave in a "random-like" fashion and their solution trajectories can deny "long-term predictability" even if the initial conditions are practically known [29, 54, 76, 109]. Such behavior is now termed *chaos,* which underlies the *complexity* and subtle *order* exhibited by real-world systems. Scientists, mathematicians and engineers from a diverging range of disciplines have found remarkably similar complex behavior in their systems. The root cause of such complex behavior has been identified collectively as *nonlinearity.* Precisely, without exception, all systems in the real world are nonlinear. In this book, we are concerned with a particular class of engineering systems, known as power electronics, which by virtue of its rich nonlinearity exhibits a variety of complex behavior.

In this introductory chapter we will take a quick tour of power electronics circuits and dynamical systems. Our aim is to introduce the basic types of switching converters, their salient operating features, modeling approaches and nonlinear behavior. We will also introduce some basic concepts of nonlinear dynamics that are necessary for understanding the complex behavior of switching converters to be described in the later chapters.

## 1.1 Overview of Power Electronics Circuits

The basic operation of any power electronics circuit involves toggling among a set of linear or nonlinear circuit topologies, under the control of a feedback system [33, 78, 81, 99, 100, 118, 128]. As such, they can be regarded as *piecewise switched* dynamical systems. For example, in simple switching converters, such as the ones shown in Figure 1.1, an inductor (or inductors) is/are "switched" between the input and the output through an appropriate switching element (labelled as $S$ in the figure). The way in which the inductor(s) is/are switched determines the output voltage level and transient behavior. Usually, a semiconductor switch and a diode are used to implement

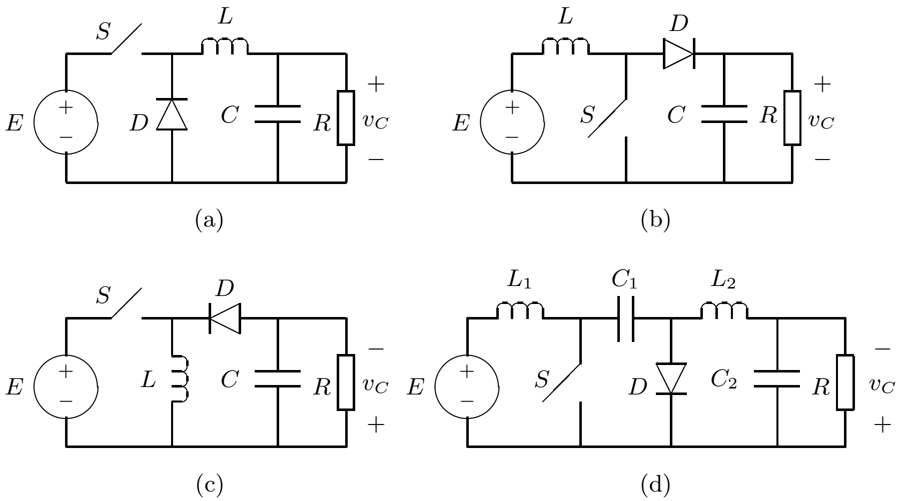



**FIGURE 1.1**
Examples of simple switching converters. (a) Buck converter; (b) boost converter; (c) buck-boost converter; (d) boost-buck (Ćuk) converter.

such switching. Through the use of a feedback control circuit, the relative durations of the various switching intervals are continuously adjusted. Such feedback action effectively controls the transient and steady-state behaviors of the circuit. Thus, both the circuit topology and the control method determine the dynamical behavior of a power electronics circuit.

### 1.1.1 Switching Power Converters

Most power converters are constructed on the basis of the simple converters shown in Figure 1.1 [128]. Typically, the switch and the diode are turned on and off in a cyclic and complementary manner. The switch is directly controlled by a pulse-width modulated signal which is derived from a feedback circuit. The diode turns on and off depending upon its terminal condition. When the switch is closed, the diode is reverse biased and hence open. Under this condition, the inductor current ramps up. When the switch is turned off, the diode is forward biased and behaves as a short circuit. This causes the inductor current to ramp down. The process repeats cyclically. The system can therefore be plainly described by a set of state equations, each responsible for one particular switch state. For the operation described above, we have two state equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{B}_1 E \quad \text{switch on and diode off} \tag{1.1}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_2\boldsymbol{x} + \boldsymbol{B}_2 E \quad \text{switch off and diode on} \tag{1.2}$$

where $\boldsymbol{x}$ is the state vector usually consisting of all capacitor voltages and inductor currents, the $\boldsymbol{A}$'s and $\boldsymbol{B}$'s are the system matrices, and $E$ is the input voltage. Furthermore, because the conduction of the diode is determined by its own terminal condition, there is a possibility that the diode can turn itself off even when the switch is off. This happens when the diode current becomes zero and is not permitted to reverse its direction. In the power electronics literature, this operation has been termed *discontinuous conduction mode,* as opposed to *continuous conduction mode* where the switch and the diode operate strictly in a complementary fashion.* Clearly, we have another state equation for the situation where both switch and diode are off.

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_3\boldsymbol{x} + \boldsymbol{B}_3 E \quad \text{switch off and diode off.} \tag{1.3}$$

In practice, the choice between continuous and discontinuous conduction modes of operation is often an engineering decision. Continuous conduction mode is more suited for high power applications, whereas discontinuous conduction mode is limited to low power applications because of the relatively high device stresses. On the other hand, discontinuous conduction mode gives a more straighforward control design and generally yields faster transient responses. Clearly, a number of factors determine whether the converter would operate in continuous or discontinuous conduction mode. For instance, the size of the inductance determines how rapidly the current ramps up and down, and hence is a determining factor for the operating mode. We will postpone the detailed discussion of the operating modes to Chapter 3.

We now examine the control of switching converters. First, as in all control systems, a control input is needed. For switching converters, the usual choice is the *duty cycle,* $d$, which is defined as the fraction of a repetition period, $T$, during which the switch is closed, i.e.,

$$d = \frac{t_c}{T} \tag{1.4}$$

where $t_c$ is the time duration when the switch is held closed. In practice, the duty cycle is continuously controlled by a feedback circuit that aims to maintain the output voltage at a fixed level even under input and load variations. In the steady state, the output voltage is a function of the duty cycle and the input voltage. For the buck converter operating in continuous conduction mode, for example, the volt-time balance for the inductor requires that the following be satisfied in the steady state:

$$(E - V_C)DT = V_C(1 - D)T \quad \Rightarrow \quad V_C = DE \quad \text{(buck converter)} \tag{1.5}$$

where uppercase letters denote steady-state values of the respective variables. Likewise, for the other converters shown in Figure 1.1 operating in continuous

*For simplicity, we omit details of the other operating modes which can possibly happen in the Ćuk converter [143].

conduction mode, we have

$$V_C = \frac{E}{1-D} \qquad \text{(boost converter)} \tag{1.6}$$

$$V_C = \frac{ED}{1-D} \qquad \text{(buck-boost converter)} \tag{1.7}$$

$$V_C = \frac{ED}{1-D} \qquad \text{(Ćuk converter)} \tag{1.8}$$

Thus, we see that as long as the duty cycle and input voltage are fixed, the output voltage will converge to a value given in the above formulas. Moreover, in the event of a transient in the load or the input voltage, the output voltage will experience a corresponding transient before it settles back to the steady-state value. Furthermore, in the event of an input voltage shift, the duty cycle value must be changed accordingly if the same output voltage is to be maintained. Clearly, we need a control circuit for output voltage regulation.

We may imagine that the simplest feedback method compares the output voltage with a reference and sends a control signal to adjust the duty cycle so as to minimize the error. Alternatively, a full state feedback can be considered. For instance, in the second-order buck, boost and buck-boost converters, both the output voltage and the inductor current can be used by the feedback circuit. In practice, two particular implementations have become the industry standard for controlling switching converters, namely, voltage feedback control and current-programmed control, also known as *voltage-mode* and *current-mode* control, respectively [83]. The former uses only the output voltage in the feedback process, and the latter uses both the output voltage and the inductor current.

### 1.1.2 Voltage-Mode Control

A typical voltage-mode controlled buck converter is shown in Figure 1.2 (a). The key feature of this control is the presence of a feedback loop which keeps track of the output voltage variation and adjusts the duty cycle accordingly. Precisely, in this control scheme, the difference between the output voltage, $v_C$, and a reference signal, $V_{\text{ref}}$, is processed by a compensation network which generates a control signal, $v_{\text{con}}$, i.e.,

$$v_{\text{con}}(t) = g(V_{\text{ref}} - v_C) \tag{1.9}$$

where $g(.)$ is a function determined by the compensation network. This control signal effectively tells how the duty cycle has to be changed in order to give the best transient dynamics for the output voltage. In a typical implementation, this control signal is compared with a periodic ramp signal, $V_{\text{ramp}}(t)$, to generate a pulse-width modulated signal which drives the switch. The ramp signal typically takes the form:

$$V_{\text{ramp}}(t) = V_L + (V_U - V_L)\left(\frac{t}{T} \bmod 1\right), \tag{1.10}$$

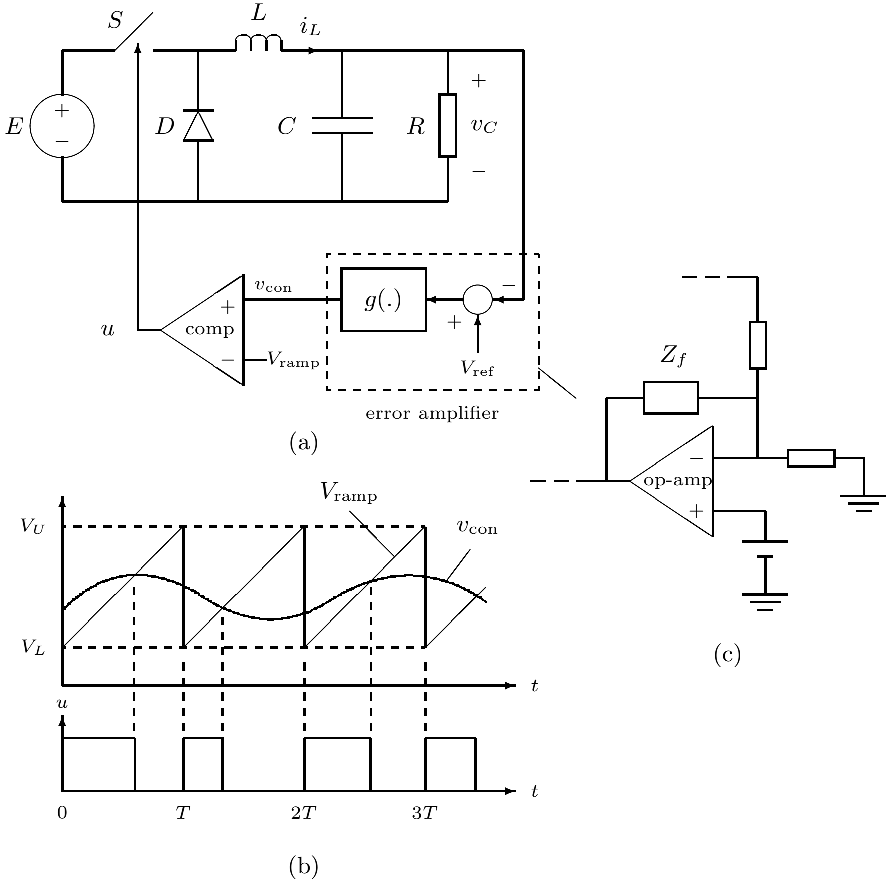



**FIGURE 1.2**
Voltage-mode controlled buck converter. (a) Circuit schematic; (b) waveforms of control signal and ramp signal; (c) possible implementation of error amplifier.

where $V_L$ and $V_U$ are the lower and upper thresholds of the ramp signal. Figure 1.2 (b) shows the interaction of the control signal and the ramp signal. Suppose the control signal moves in the opposite direction as the output voltage, i.e., $v_{\rm con}$ goes up when the output voltage decreases, and vice versa. Then, the output voltage can be regulated with the following switching rule:

$$\text{Switch} = \begin{cases} \text{on} & \text{if } V_{\rm ramp}(t) \le v_{\rm con}(t) \\ \text{off} & \text{if } V_{\rm ramp}(t) > v_{\rm con}(t) \end{cases} \tag{1.11}$$

which can be easily implemented by a comparator, as shown in Figure 1.2 (a). Thus, the duty cycle at the $n$th switching period, $d_n$, is given implicitly by

$$v_{\text{con}}((d_n + n)T) = V_{\text{ramp}}((d_n + n)T). \tag{1.12}$$

We can easily verify in this case that if the control signal goes up as a result of an output voltage drop, the duty cycle increases.* Thus, the feedback action regulates the output voltage, and the closed-loop dynamics can be shaped by the compensation network.

### 1.1.3 Current-Mode Control

For current-mode control, an inner current loop is used in addition to the voltage feedback loop. The aim of this inner loop is to force the inductor current to follow some reference signal provided by the output voltage feedback loop. The result of current-mode control is a faster response. This kind of control is mainly applied to boost and buck-boost converters which suffer from an undesirable non-minimum phase response [83, 128]. A simplified schematic is shown in Figure 1.3 (a). The circuit operation of the inner loop can be described as follows. Suppose the switch is now turned on by a clock pulse. The inductor current thus rises up, and as soon as it reaches the value of the reference current $I_{\text{ref}}$, the comparator output goes momentarily high and turns off the switch. The inductor current then ramps down. The process repeats as the next clock pulse turns the switch back on. Figure 1.3 (b) describes the typical inductor current waveform. By inspecting the waveform, we can write the duty cycle at the $n$th switching period implicitly as

$$d_n = \frac{I_{\text{ref}}((d_n + n)T) - i_L(nT)}{(E/L)T} \tag{1.13}$$

To achieve output voltage regulation, an output voltage loop is needed, as shown in Figure 1.3 (a). This loop senses the output voltage error and adjusts the value of $I_{\text{ref}}$ accordingly. In practice, the inner current loop is a much faster loop compared to the output voltage loop. Thus, when we study the inner current loop dynamics, we may assume that $I_{\text{ref}}$ is essentially constant or varying slowly. Details of the analysis of this system are left to Chapter 5.

With the inductor current taken into account, current-mode control generally performs better. In practice, however, the application of current-mode control to the buck converter does not gain much benefit over voltage-mode control. This is because the inductor current information can be readily derived from the output voltage in the case of the buck converter. Thus, with

*Depending on how the error amplifier is connected, the control voltage can be designed to react in the same or opposite direction as the output voltage. If the control signal goes in the same direction as the output voltage, the switching rule (1.11) must be reversed in order to regulate the output voltage. The choice is arbitrary.

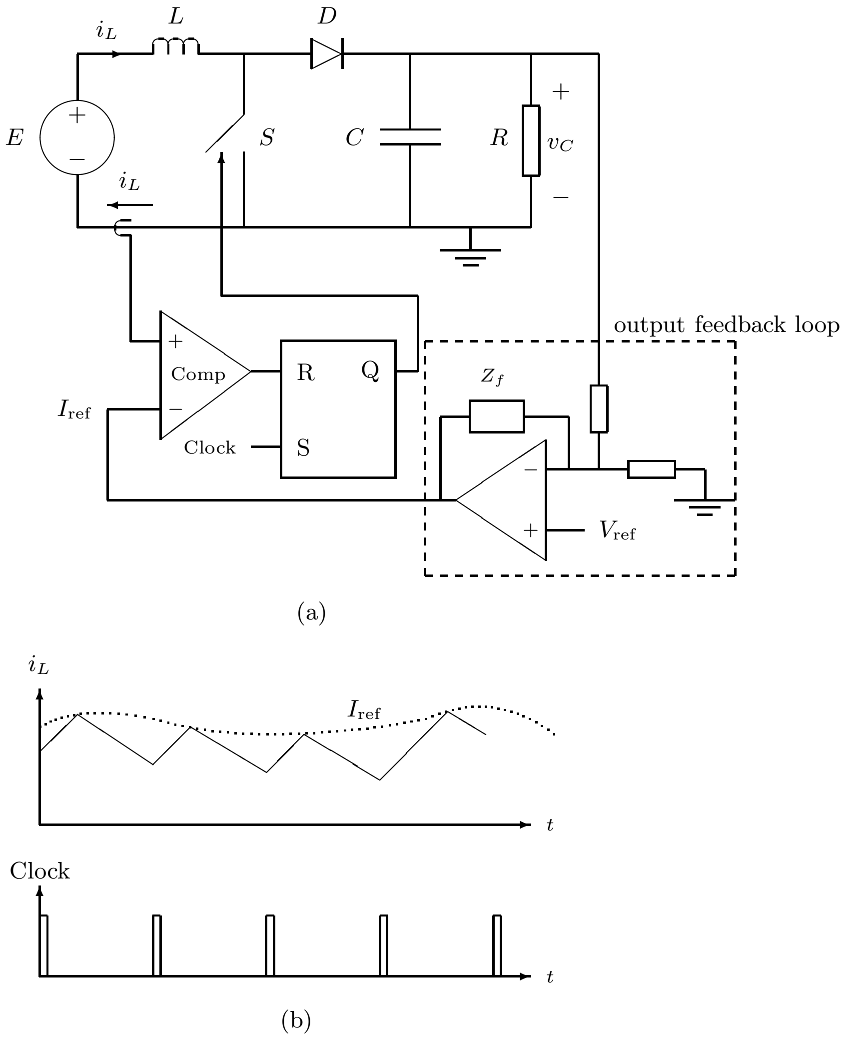



**FIGURE 1.3**
Current-mode controlled boost converter. (a) Circuit schematic; (b) waveforms of inductor current and reference current.

an appropriate design of the compensation circuit, voltage-mode control can achieve comparable performance as current-mode control. When applied to the boost or buck-boost converter, the benefits of current-mode control becomes significant. Essentially, since the inductor current is programmed to follow a reference current (which is in turn derived from the output voltage), its averaged dynamics is "destroyed." Thus, for frequencies much below the switching frequency, the inductor current dynamics becomes insignificant, making the design of the compensator much easier to perform. Besides, with the absence of the low-frequency inductor current dynamics, the inherent non-minimum phase problem associated with the boost and buck-boost converters is automatically eliminated. However, current-mode control is not completely free from stability problems. In fact, it has been shown that high-frequency instability in the form of subharmonics and chaos is possible in current-mode controlled converters, as will be detailed in Chapter 5.

### 1.1.4 Complexity of Operation

Up till now, switching power converters have always been designed to operate in only one specific type of periodic operation, commonly known as *period-1* operation, in which all waveforms repeat at the same rate as the driving clock. Most converter circuits are thus expected to work stably in this regime under all possible disturbances. However, period-1 operation is not the only possibility. For instance, under certain conditions, the circuit may operate in a period-$n$ regime in which the periods of all waveforms are exactly $n$ times that of the driving clock. We can immediately appreciate the complexity in the operation of switching converters, where a variety of operational regimes exist and a large number of parameters may affect the stability of a particular regime. As parameters vary, the operation can go from one regime to another, sometimes in an abrupt manner. Such a phenomenon, where one regime fails to operate (e.g., as a result of a loss of stability) and another one picks up, is termed *bifurcation.** Thus, even when a converter is well designed to work in a particular (desired) regime, it could fail to operate as expected if some parameters are varied, causing it to assume another regime. If the newly assumed regime is an undesirable one, locating the bifurcation boundary becomes imperative. A few basic questions are often posed to the engineers:

1. What determines the operating regime of a given system?
2. How can we guarantee that a circuit operates in a desired regime?
3. When a system fails to operate in its desired operating regime, what is then the operating regime it would assume?

To answer these questions, we need to develop appropriate simulation and experimental tools (see Chapter 2). We also need to derive appropriate models to facilitate analysis (see Section 1.2 and Chapter 3). Most importantly, we have to identify the basic phenomenology associated with each system under study. For nonlinear systems, there is no stereotypical result that fits all. We have to tackle each system separately.

*Bifurcation literally means splitting into two parts. In nonlinear dynamics, the term has been used to mean splitting of the behavior of a system at a threshold parameter value into two qualitatively different behaviors, corresponding to parameter values below and above the threshold [65].

## 1.2 Overview of Modeling Strategies for Switching Converters

As mentioned before, switching converters are essentially piecewise switched circuits. The number of possible circuit topologies is usually fixed, and the switching is done in a cyclic manner (but not necessarily periodically because of the feedback action). This results in a nonlinear time-varying operating mode, which naturally demands the use of nonlinear methods for analysis and design.

### 1.2.1 From Nonlinear Models to Linear Models

Power electronics engineers are always dealing with nonlinear problems and have attempted to explore methods not normally used in other circuit design areas, e.g., state-space averaging [98], phase-plane trajectory analysis [108], Lyapunov based control [126], Volterra series approximation [159], etc. However, in order to expedite the design of power electronics systems, "adequate" simplifying models are imperative. In the process of deriving models, accuracy is often traded off for simplicity for many good practical reasons. Since closed-loop stability and transient responses are basic design concerns in practical power electronics systems, models that can permit the direct application of conventional small-signal approaches will present obvious advantages. Thus, much research in modeling power electronics circuits has been directed toward the derivation of linearized models that can be applied in a small-signal analysis, the limited validity being the price to pay. (The fact that most engineers are trained to use linear methods is also a strong motivation for developing linearized models.) The use of linearized models for analysis is relatively mature in power electronics. However, it falls short of predicting any nonlinear behavior.

### 1.2.2 Back to Nonlinear Models

Since our purpose here is nonlinear analysis, we will not consider linearization right at the start of the analysis, which effectively suppresses all nonlinear terms. In fact, linearization is a useful technique only when we need to characterize the system behavior locally around a point in the state space. The major modeling step prior to linearization is the derivation of a suitable nonlinear model. In this book we will focus on two particularly useful modeling approaches:

1. Continuous-time averaging approach
2. Discrete-time iterative mapping approach (or simply discrete-time approach)

*Averaging Approach*

Probably the most widely adopted modeling approach for switching converters is the averaging approach which was developed by R.D. Middlebrook in the 1970s [98]. This modeling approach effectively removes the time-varying dependence from the original time-varying model. The ultimate aim is to produce a continuous-time state equation which contains no time-varying terms. The key idea in this approach lies in discarding the switching details of the state variables and retains only their "average" dynamics. In the modeling process, the state equations corresponding to all possible stages are first written down, and the final model is simply the weighted average of all the state equations. The weightings are determined from the relative durations of the stages. Typically, an averaged model takes the form:

$$\frac{d\boldsymbol{x}}{dt} = \left( \sum_{i=1}^{N} d_i \boldsymbol{A}_i \right) \boldsymbol{x} + \left( \sum_{i=1}^{N} d_i \boldsymbol{B}_i \right) E \tag{1.14}$$

where $\boldsymbol{x}$ is the state vector, $N$ is the number of stages in a period, $d_i$ is the fractional period (duty cycle) of the $i$th stage, $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ are the system matrices for the $i$th stage. Finally, we need to state the control law in order to complete the model. This is usually given as a set of equations defining explicitly or implicitly the quantities $d_j$. The general form of such a set of equations is

$$\begin{cases} G_1(d_1, d_2, ..., E, \boldsymbol{x}) = 0 \\ G_2(d_1, d_2, ..., E, \boldsymbol{x}) = 0 \\ \cdots \end{cases} \tag{1.15}$$

Note that the above equations generally define the duty cycles $d_j$ as nonlinear functions of the system states and parameters. Thus, despite its appearance, the averaged model is nonlinear. Clearly, the averaged model so derived has left out all high-frequency details, and hence is not suitable for characterizing high-frequency or fast-scale dynamics. As a rule, we should only use an averaged model for analysis or characterization of phenomena which occur as fast as an order of magnitude below the switching frequency.

*Discrete-Time Mapping Approach*

Another modeling approach that provides fuller dynamical information is the discrete-time iterative mapping approach. Here, we aim to model the dynamics in a discrete manner. We take the value of the state vector at the start of a period, say $\boldsymbol{x}_n$, follow its trajectory through all the $N$ stages, and find its value at the end of the period. The ultimate aim is to produce a difference equation of the form:

$$\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n, \boldsymbol{d}, E) \tag{1.16}$$

where $\boldsymbol{x}_n$ is the state vector at $t = nT$, $E$ is the input voltage, $\boldsymbol{d}$ is the vector of the duty cycles, i.e., $\boldsymbol{d} = [d_1 \;\; d_2 \;\; \cdots \;\; d_N]^T$. To complete the model, a control equation similar to (1.15) is needed. It is worth noting that the above description assumes the sampling period be equal to the switching period. Thus, the model so obtained is capable of describing the dynamical variation up to the switching frequency.

Needless to say, the two modeling approaches have their own advantages and disadvantages. Intuitively, the averaged model should be quite easy to obtain (involving less algebraic manipulation) whereas the discrete-time iterative model would probably involve more tedious algebra. They also deviate in their capabilities of characterizing dynamical behavior of a given system. Generally speaking, the averaged model is good for slow-scale (low-frequency) characterization whereas the discrete-time model is good for fast-scale (high-frequency) characterization. In Chapter 3, we will take a detailed look at the modeling processes and their capabilities.

## 1.3 Overview of Nonlinear Dynamical Systems

As we have seen in the foregoing section, switching power converters can be modeled by a continuous-time differential equation or a discrete-time difference equation. In general, any system that can be put in such a form is a dynamical system in the sense that its behavior varies as a function of time [44, 55, 56]. More precisely, what constitutes a dynamical system is

- a set of independent state variables; and
- a function which connects the rates of change of the state variables with the state variables themselves and other inputs.

In an electrical circuit, for example, the inductor currents and capacitor voltages form a set of independent state variables.* The basic constitutive laws of all elements (i.e., $v = iR$ for resistors, $L(di/dt) = v$ for inductors, $C(dv/dt) = i$ for capacitors, and other possible nonlinear laws), together with the relevant independent Kirchhoff's law equations, give the connecting function [144]. Thus, with a set of state variables and a connecting function, we can describe a dynamical system. Further, we may assume that the following form is universal for describing a dynamical system:

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t), \mu, t) \tag{1.17}$$

*We emphasize "independent" here. If a circuit contains dependent inductor currents and/or capacitor voltages, the number of state variables should be less than the number of inductors and capacitors. In the circuit theory literature, there are well established rules to identify independent state variables. See for example the texts by Rohrer [124] and Tse [144].

where $\boldsymbol{x}$ is the vector consisting of the state variables, $\boldsymbol{f}$ is the connecting function, and $\mu$ is a vector of parameters. The above system, with $\boldsymbol{f}$ being dependent upon time, is called a *non-autonomous* system. Moreover, if the time dependence is absent in $\boldsymbol{f}$, i.e.,

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t), \mu), \tag{1.18}$$

the system is *autonomous.*

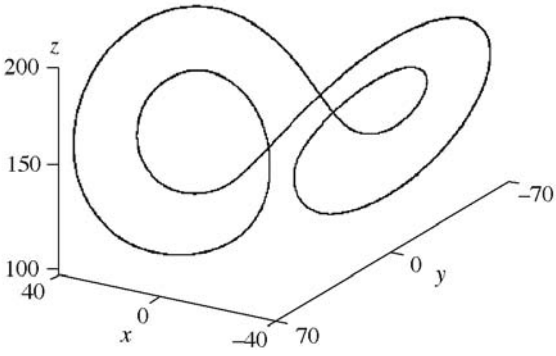
In switching converters, distinction between non-autonomous and autonomous systems can be made conveniently by the presence or absence of a fixed frequency driving clock. In the past, most converters were constructed in a free-running mode, typically using a hysteretic or self-oscillating control circuit. Such systems are therefore autonomous. Nowadays, with the advent of integrated circuits (ICs), fixed frequency oscillators are easily implemented and most switching converters are designed to operate periodically under a fixed frequency clock which comes with most control ICs. Such systems are therefore non-autonomous. For example, the circuits shown in Figures 1.2 and 1.3 are non-autonomous systems.

### 1.3.1 Qualitative Behavior of Dynamical Systems

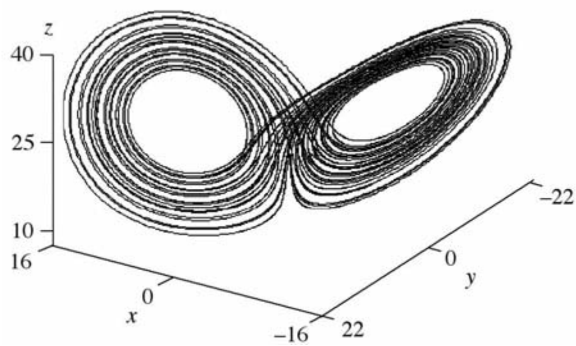
The afore-described dynamical systems are often called *deterministic systems,* in the sense that the exact way in which they evolve as time advances is fully determined by the describing differential equations [4, 53]. Precisely, given an initial condition, the solution of the system, also known as the *trajectory,* is completely determined. For linear systems, we know that closed-form solutions can be found. But for nonlinear systems, closed-form solutions are almost always unavailable, and numerical solutions must be sought.

After an initial transient period, the system soon enters its steady state. The solution in the steady state can be regarded as an *equilibrium solution,* in the sense that if the system starts at a point on this solution, it stays permanently on that solution. Thus, we may conceive that there could be many equilibrium solutions which may or may not be steady-state solutions. When the system is let go from a point outside these equilibrium solutions, it converges to only one of them. The equilibrium solution to which the system converges is called an *attracting equilibrium solution* or simply an *attractor.* In nonlinear systems, the behavior can be further complicated by the selective convergence to an equilibrium solution depending upon the initial point. In other words, there may be two or more competing attractors, and depending on the initial condition, the system converges selectively to one of them. Thus, to determine the steady-state behavior of a system, we have to know the possible attractors as well as their respective *basins of attraction.*

For ease of visualization, we refer to a 3-dimensional state space in the following discussion of attractors. In general, we may classify attractors under the following categories:




(a)




(b)

**FIGURE 1.4**
Attractors from the Lorenz system [134]: $\dot{x} = 10(y - x)$, $\dot{y} = -xz + rx - y$ and $\dot{z} = xy - 8z/3$. (a) Limit cycle with $r = 160$; (b) chaotic attractor with $r = 25$.

1. *Fixed point:* The solution is a point in the state space.

2. *Limit cycle or periodic orbit:* The trajectory moves along a closed path in the state space. Furthermore, this motion is associated with a finite number of frequencies, which are related to one another by rational ratios. The motion is periodic. An example is shown in Figure 1.4 (a).

3. *Chaotic attractor:* The trajectory appears to move randomly in the state space. Moreover, the trajectory is bounded and the motion is non-periodic. An example is shown in Figure 1.4 (b). We will discuss the properties of chaos in more detail in Section 1.3.3.

4. *Quasi-periodic orbit:* The trajectory moves on the surface of a torus, as illustrated in Figure 1.5. The motion is associated with a finite number

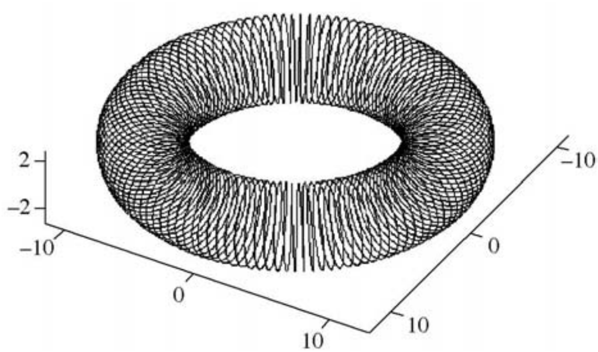



**FIGURE 1.5**
Quasi-periodic orbit. The trajectory moves on the surface of the torus and eventually visits every point on that surface. The motion is characterized by two rotations, one around the large circumference at frequency $f_1$ and the other around the cross section of the torus at frequency $f_2$. The ratio of $f_1$ to $f_2$ is irrational.

of frequencies, which are related to one another by irrational ratios. The motion appears "almost periodic" but is not exactly periodic.

### 1.3.2 Bifurcation

As mentioned before, a dynamical system can have multiple equilibrium solutions. For a given set of parameters and initial condition, the system converges to one of the equilibrium solutions. This equilibrium solution is the attractor. If the parameters are allowed to vary, the system may relinquish its presently assumed equilibrium solution and pick up another equilibrium solution. For instance, as the parameters vary, the presently assumed equilibrium solution becomes unstable and the system is attracted to another stable equilibrium solution. This phenomenon is termed *bifurcation,* as we have briefly mentioned before. In general, bifurcation can be regarded as a sudden change of qualitative behavior of a system when a parameter is varied. We may therefore classify bifurcation according to the type of qualitative change that takes place when a parameter is varied. In the following we briefly summarize some commonly observed bifurcations in physical and engineering systems [1, 2, 3, 65, 85, 104, 109].

1. *Saddle-node bifurcation:* This type of bifurcation is characterized by a sudden loss or acquisition of a stable equilibrium solution as a parameter moves across a critical value. Systems that exhibit a saddle-node bifurcation can be "normalized" to the form $\dot{x} = \mu \pm x^2$, where $\mu$ is the

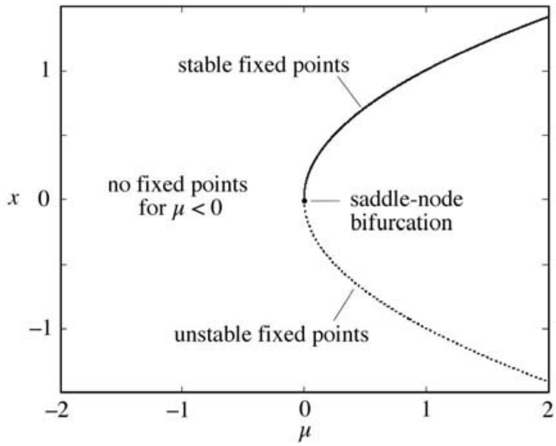



**FIGURE 1.6**
Saddle-node bifurcation of the system $\dot{x} = \mu - x^2$. As $\mu$ goes from negative to positive, a stable fixed point suddenly appears. Conversely, as $\mu$ goes from positive to negative, the stable fixed point suddenly disappears.

parameter and its critical parameter value is 0.* Figure 1.6 illustrates this bifurcation.

2. *Transcritical bifurcation:* This type of bifurcation is characterized by an exchange of stability status of two equilibrium solutions, as illustrated in Figure 1.7. Precisely, the system initially has one stable equilibrium solution and one unstable equilibrium solution. As a parameter is varied and reaches a critical value, the stable equilibrium solution becomes unstable, while the unstable equilibrium one becomes stable and takes over. The form of the system equation that exhibits a transcritical bifurcation can be normalized to $\dot{x} = \mu x \pm x^2$. The critical value of $\mu$ is again 0.

3. *Supercritical pitchfork bifurcation:* This type of bifurcation is characterized by splitting of a stable equilibrium solution into two stable equilibrium solutions at the critical parameter value. Precisely, the system exchanges stability status between one equilibrium solution and another pair of equilibrium solutions. Systems exhibiting this type of bifurcation

*From the center manifold theorem [53, 77, 138], any local bifurcation of an $N$-dimensional system can be analyzed by examining the so-called center manifold at the point of bifurcation, which is an $M$-dimensional ($M < N$) subspace tangential to the eigenspace corresponding to zero eigenvalue(s) of the Jacobian evaluated at the bifurcation point. The normalized system equation shown above describes the dynamics on this center manifold.

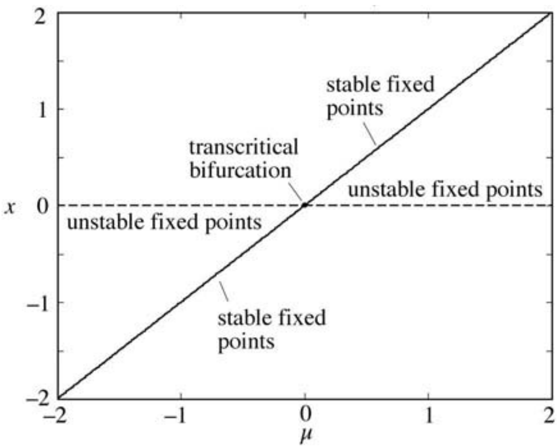



**FIGURE 1.7**
Transcritical bifurcation of the system $\dot{x} = \mu x - x^2$. As $\mu$ moves across zero, stability suddenly exchanges between two fixed points.

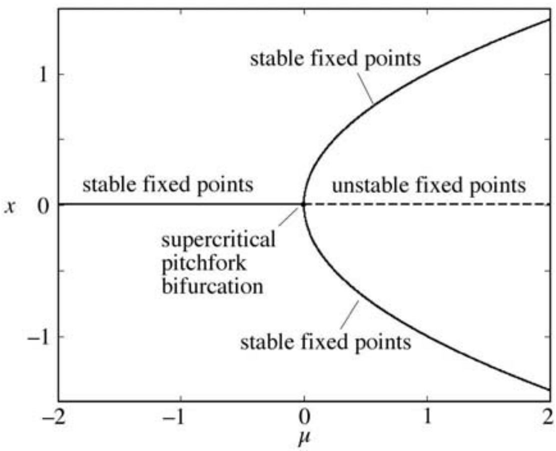



**FIGURE 1.8**
Supercritical pitchfork bifurcation of the system $\dot{x} = \mu x - x^3$. As $\mu$ goes from negative to positive, the stable fixed point suddenly forks off into two stable fixed points. The system is then attracted to one of the stable fixed points.

can be normalized to the form $\dot{x} = \mu x - x^3$, where $\mu = 0$ is the critical parameter value. Figure 1.8 illustrates this bifurcation.

4. *Subcritical pitchfork bifurcation:* This type of bifurcation is characterized by a sudden explosion of a stable equilibrium solution as a pa-

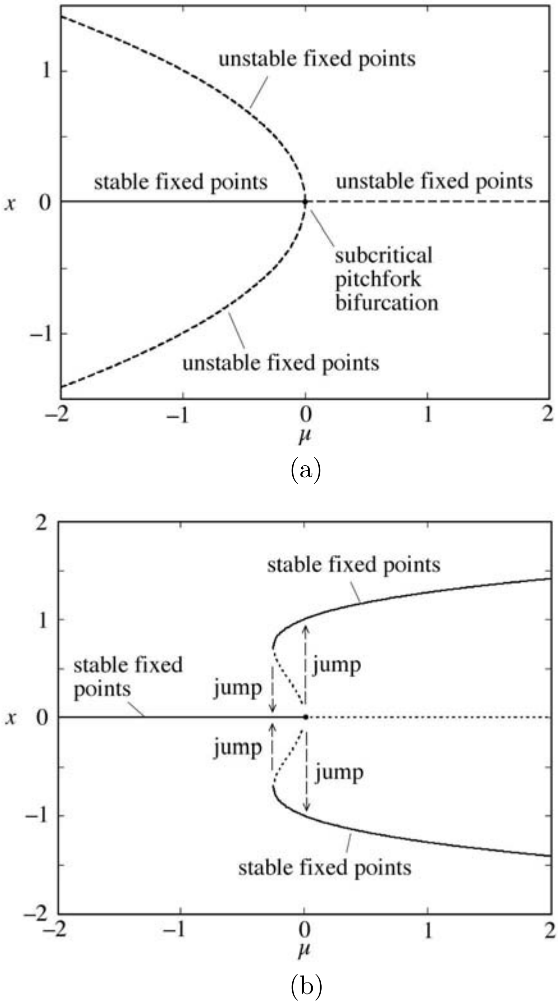



**FIGURE 1.9**
(a) Subcritical pitchfork bifurcation of the system $\dot{x} = \mu x + x^3$. As $\mu$ goes from negative to positive, the stable fixed point suddenly blows up; (b) sudden "jump" in real systems due to the presence of higher order terms $\dot{x} = \mu x + x^3 - x^5$. Note that a hysteresis loop exists. When $\mu$ moves in backward direction, the jump occurs at a negative value of $\mu$.

rameter moves across a critical value. The normalized equation takes the form of $\dot{x} = \mu x + x^3$, where $\mu = 0$ is the critical parameter value. Figure 1.9 (a) illustrates this bifurcation. In real systems, higher order terms always exist to counteract the explosion, e.g., $\dot{x} = \mu x + x^3 - x^5$.

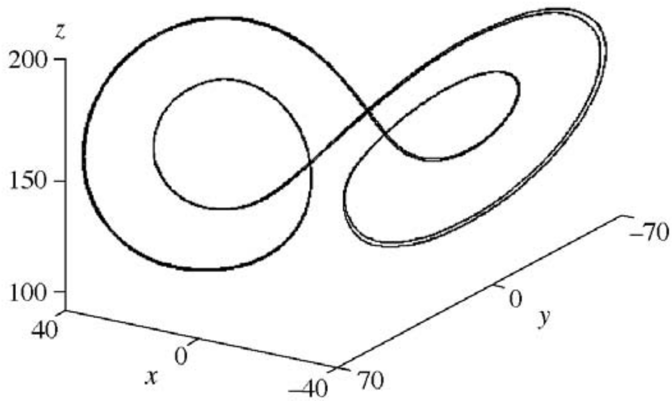



**FIGURE 1.10**
Period-2 orbit with $r = 149$ in the Lorenz system (see caption of Figure 1.4).

In this case, the system does not blow up at $\mu = 0$, but "jumps" to another stable equilbrium solution, as illustrated in Figure 1.9 (b).

5. *Period-doubling bifurcation:* This type of bifurcation is characterized by a sudden doubling of the period of a stable periodic orbit or limit cycle. Using the example of the Lorenz system shown in Figure 1.4, we may observe period-doubling bifurcation by varying the parameter $r$. Specifically, the periodic orbit shown in Figure 1.4 loses stability when $r$ is decreased to around 149, and at that point, a period-2 orbit takes over, as shown in Figure 1.10. Further decreasing $r$ to about 147, the period doubles again, as shown in Figure 1.11.

6. *Hopf bifurcation:* This type of bifurcation is characterized by a sudden expansion of a stable fixed point to a stable limit cycle. Systems that exhibit this bifurcation can be normalized to a second-order equation of the form $\dot{x} = -y + x[\mu - (x^2 + y^2)]$, $\dot{y} = x + y[\mu - (x^2 + y^2)]$. For $\mu < 0$, the system has a stable fixed point ($x = y = 0$), which is associated with a pair of complex eigenvalues having negative real parts. As $\mu$ goes from negative to positive, the pair of complex eigenvalues move across the imaginary axis, i.e., the real parts become positive. Thus, the fixed point loses stability. However, due to the second-order terms, the system has a stable limit cycle of radius $\sqrt{\mu}$ for $\mu > 0$.

7. *Border collision:* This type of bifurcation occurs in dynamical systems where two or more structurally different systems operate for different parameter ranges. When a parameter is varied across the boundary of two structurally different systems, an abrupt change in behavior occurs.

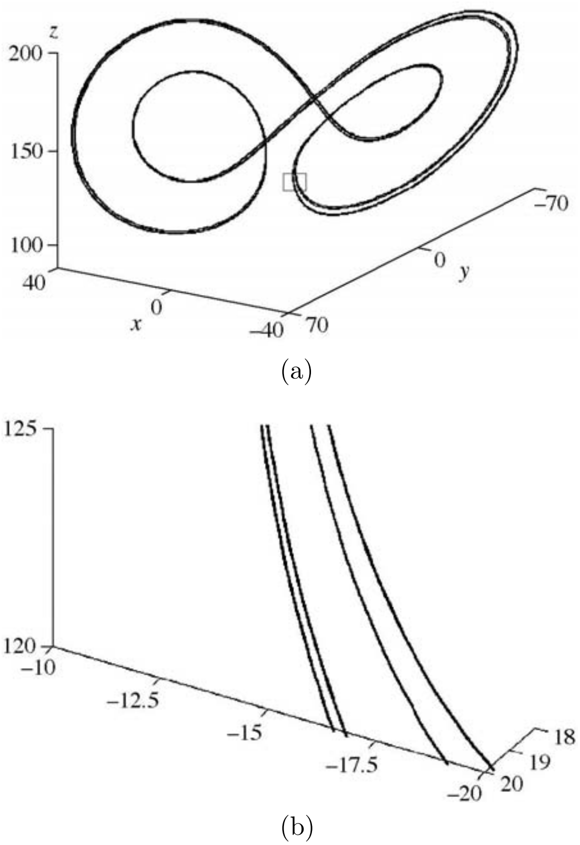



**FIGURE 1.11**
(a) Period-4 orbit with $r = 147$ in the Lorenz system; (b) enlargement of the small framed area.

> This is known as border collision. The exact type of behavioral change depends on the dynamics of the systems corresponding to the two sides of the boundary.

It is worth noting that with the exception of border collision, the afore-described types of bifurcation do not involve structural changes of the system. They are sometimes called *smooth bifurcation* or *standard bifurcation.* The meaning of the adjective "smooth" has a mathematical origin, which relates to the differentiability of the function that describes the system. Coincidentally, the term "non-smooth" fits well with the appearance of the bifurcation diagrams which manifest rather unusual transitions not resembling

**TABLE 1.1**
Qualitative differences between "smooth" (standard) bifurcations and border collisions.

| Characteristics | "Smooth" (standard) bifurcations e.g., period-doubling, Hopf, etc. | Border collisions |
|---|---|---|
| Cause | Loss of stability | Alteration of circuit operation |
| Structure of system | Structurally unchanged (topological sequence unchanged) | Structurally changed (topological sequence altered) |
| Manifestation in bifurcation diagrams | Appearance as typified in bifurcation diagrams of standard types | Abrupt transitions not resembling any standard bifurcation (e.g., abrupt bendings, discontinuities, jumps) |

those in standard bifurcations, as we will see in later chapters.* Furthermore, a "smooth" bifurcation is normally associated with the loss of stability of one solution and the picking up of another, whereas border collision is characterized by abrupt alteration of the detailed operating principle. In other words, a "smooth" bifurcation occurs at a stability boundary, whereas border collision occurs at an operation boundary where the system experiences an operational change. We will discuss what we mean by an operational change more precisely in Section 1.4. Table 1.1 summarizes the basic differences between these two classes of bifurcations.

### 1.3.3 Deterministic Chaos

As mentioned earlier in Section 1.3.1, chaos is a particular qualitative behavior of nonlinear systems, which is characterized by an aperiodic and apparently random trajectory [115]. In addition, the trajectory is unpredictable in the long term, meaning that knowing the trajectory at this time gives no information about where exactly the trajectory will be in the far future. Note that the dynamics of any deterministic system can be theoretically described by differential equations, although the derivation of such differential equations may prove to be difficult for very complicated systems. A classic example of an apparently random system is the flipping of a coin. The final outcome, either a head or a tail, appears to be unpredictable. However, the process of

*Here, we may regard bifurcation diagrams as summary charts of behavioral changes, which typically record the change of behavior of a system as some parameter(s) is/are varied.

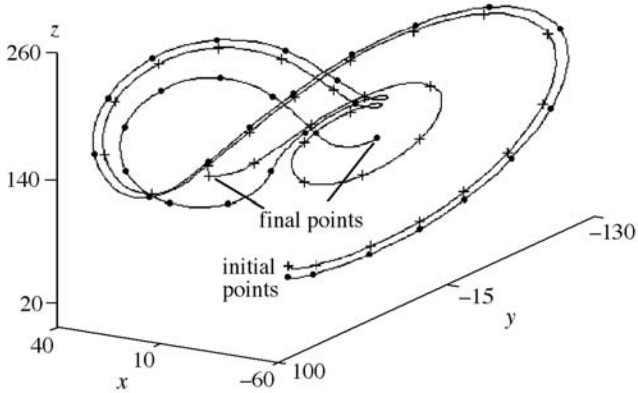



**FIGURE 1.12**
Two trajectories of the Lorenz system: $\dot{x} = 10(y - x)$, $\dot{y} = -xz + 25x - y$ and $\dot{z} = xy - 8z/3$. At $t = 0$, the trajectory labelled with "+" starts at (0, –5, 15), and the one labelled with "•" starts at (1, –6, 16). The final points are taken at $t = 1$. Note that if the initial points are set closer, a longer time is needed to observe the divergence of the two trajectories.

generating any particular outcome in this system is unarguably deterministic. First, the initial position of the coin can theoretically be known. Then, the initial velocity, gravitational force, air viscosity, the mass and moment of inertia of the coin, etc. are all theoretically known or knowable. Therefore, deterministic equations can be theoretically written to describe the motion of the coin as it is thrown up and later falls under the force of gravity. Finally, its landing position is also theoretically computable. The question is what makes the outcome random and unpredictable. In fact, this question is shared by all deterministic systems which exhibit apparent randomness and deny long-term predictability.

The answer to the above question lies in a key property of chaotic systems, which is now widely known as *sensitive dependence on initial condition.* In brief, two nearby starting points can evolve into two entirely uncorrelated trajectories. We take the Lorenz system again as an example, and examine two trajectories beginning at two nearby points. As shown in Figure 1.12, the two trajectories initially stay close to each other, but quickly move apart. We should now appreciate the difficulty of predicting where the system will end up eventually. In other words, the trajectory is unpredictable in the long run because there is a limit to which the starting condition can be accurately located. In our earlier example of tossing a coin, we may begin each time with a slightly different initial condition, including the position of the coin, upward velocity, spinning speed, etc. The final landing position is therefore unpredictable, even though the system is deterministic.

### 1.3.4 Quantifying Chaos

The afore-described property of being sensitively dependent upon initial condition can be taken as a defining property of chaotic systems [41, 65, 162, 167]. Thus, we may test whether a system is chaotic by evaluating its sensitivity to a change of initial condition. To illustrate how the sensitivity to initial condition can be quantified, we consider a first-order system which is defined by

$$\dot{x} = f(x). \tag{1.19}$$

Suppose $x_0(t)$ is the trajectory corresponding to an initial value $x_0$. We consider another trajectory which starts at a nearby point, say $x_0 + \epsilon_0$. We simply denote this trajectory by $x(t)$. Clearly, what we are interested in is the difference between $x(t)$ and $x_0(t)$ as time elapses. Let this difference be $s(t)$, i.e.,

$$s(t) = x(t) - x_0(t). \tag{1.20}$$

If we assume that $s(t)$ grows exponentially, we may write $s(t) = s(0)e^{\lambda t}$, where $\lambda$ can be found empirically to fit the divergence rate. Alternatively, we may describe the dynamics of $s(t)$ by

$$\dot{s} = \lambda s. \tag{1.21}$$

Moreover, the Taylor's expansion of $f(x)$ around $x_0$ is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots \tag{1.22}$$

Thus, ignoring higher-order terms in (1.22), the variable $s(t)$ changes at a rate given by

$$\begin{aligned} \dot{s}(t) &= \frac{d}{dt}\left(x(t) - x_0(t)\right) \\ &= f'(x_0)(x - x_0). \end{aligned} \tag{1.23}$$

Now, from (1.21) and (1.23), we get

$$\lambda = f'(x_0). \tag{1.24}$$

Therefore, we may test divergence of the two trajectories, $x(t)$ and $x_0(t)$, by inspecting the sign of $\lambda$. Precisely, a positive value of $\lambda$ indicates that the two trajectories diverge at the point $x_0$, whereas a negative value indicates convergence. The quantity $\lambda$ has been known as the *Lyapunov exponent.* Furthermore, the value of the Lyapunov exponent may change along the trajectory. Thus, we need to look at the average value of the Lyapunov exponent along a sufficiently long segment of the trajectory in order to tell whether nearby trajectories diverge exponentially on the average. The test for chaos should therefore be based on the *average Lyapunov exponent.* In brief, *if the* *average Lyapunov exponent is positive, the system is sensitively dependent upon initial condition and thus is chaotic* [77].

The above concept of measuring the divergence rate of two nearby trajectories can be extended to higher-order systems. If we consider an $N$th order system, the expansion or contraction of $s(t)$ at a specific point must be associated with specific directions. In general, there should be $N$ Lyapunov exponents corresponding to $N$ directions in the state space. We note that $f'(x)$ in the above first-order system is simply the eigenvalue of the system, which describes the divergence rate of the error near $x$. In an $N$th order system, the $N$ Lyapunov exponents at a certain point are the $N$ eigenvalues evaluated at that point. Each of these Lyapunov exponents is associated with a direction of expansion or contraction which is given by the corresponding eigenvector. Thus, at a certain point, the trajectory may expand in some direction, and contract in another. If any one of the Lyapunov exponents is positive, nearby trajectories are diverging at that point. Again, we need to take the average of the Lyapunov exponents along a sufficiently long segment of the trajectory. For the higher-order case, we conclude that if the "largest" average Lyapunov exponent is positive, the system is sensitively dependent upon initial condition and thus is chaotic. In Chapter 2, we will describe the computation of the average Lyapunov exponents in some detail.

### 1.3.5 Routes to Chaos

In the foregoing we have shown that randomness and lack of predictability are the key elements of chaos. However, being random or unpredictable does not necessarily mean that no systematic study can be pursued on the complex behavior of nonlinear systems. In fact, behind the complex behavior, there is always some *subtle order* that governs the way complexity is organized. In particular, in studying chaos, we often try to find some traceable precursors so that we might tell if chaos is likely to happen in an otherwise non-chaotic system. We have seen earlier that nonlinear systems can exhibit a variety of behavior, chaos being one particular type. We have also seen that nonlinear systems can undergo bifurcation whereby qualitative behavior can change from one type to another. In the literature, the term *route to chaos* has been commonly used to refer to the series of bifurcations through which non-chaotic behavior transmutates into chaotic behavior. Here, we summarize a few important routes to chaos [109].

1. *Route to chaos via period-doubling:* As discussed earlier, some nonlinear systems may undergo period-doubling bifurcation as a certain parameter is varied. This doubling of the period may continue to occur when the same parameter is varied in the same direction. Eventually, the behavior becomes chaotic. In fact, the Lorenz system shown earlier exhibits this type of route to chaos, as the parameter $r$ is varied. We recall that when $r = 160$, the steady-state behavior is periodic (i.e., exhibiting a

limit cycle), as shown in Figure 1.4 (a). As we reduce $r$ to about 149, we observe a period-doubling bifurcation, and if we further reduce $r$ to about 147, we observe another period-doubling bifurcation. Figures 1.10 and 1.11 show the period-2 and period-4 attractors. In fact, period-doubling bifurcation continues to occur as $r$ is reduced. When $r$ is about 144, the attractor is chaotic.

2. *Route to chaos via quasi-periodicity:* Some nonlinear systems may undergo Hopf bifurcation whereby a stable fixed point changes to a limit cycle as a certain parameter is varied. As the parameter continues to vary, the system admits another periodicity which is not in a rational ratio to that of the first limit cycle. The resulting behavior is quasi-periodic. Under some circumstances, upon further varying the parameter, the behavior becomes chaotic.

3. *Route to chaos via intermittency:* Some nonlinear systems exhibit chaotic behavior intermittently, with bursts of chaotic behavior separated by long intervals of periodic behavior. Under the variation of a certain parameter, the bursts of chaotic behavior become progressively longer while the intervals of periodic behavior become shorter. Eventually, the behavior becomes fully chaotic.

4. *Crisis:* Some nonlinear systems may all of a sudden become chaotic when a certain parameter is varied. There is no traceable route to chaos in the form of a sequence of events. Crisis may be encountered, for example, when an attractor "collides" with an unstable chaotic orbit, causing the attractor to span also the unstable chaotic orbit. The result is a sudden expansion to chaos.

## 1.4 Complex Behavior in Power Electronics

Chaos and bifurcation have long been observed by power electronics engineers in the course of developing power electronics circuits [169]. Problems such as subharmonic oscillations, intermittent chaos, quasi-periodic and chaotic operations are not at all uncommon. Because of the complexity of these problems, most practicing engineers have resorted to quick fixes via some trial-and-error procedures, the aim being just to get rid of the undesirable operations. With the success of nonlinear dynamics research in the 1970s, the complex behavior in power electronics has begun to receive some formal treatments since the late 1980s, and much of the reported work has focused on switching power converters. Research in this field has now reached a point where the basic phenomena associated with some commonly used power converters have been identified. Of particular importance is the identification of bifurcation phenomena, which has played a crucial role in improving our understanding of the complex behavior exhibited by switching converters.

Power electronics can exhibit both smooth bifurcation and border collision, depending upon whether a structural change is involved. It should be noted that the switching between one topology to another during the normal operation of a power converter should not be considered as structural change (for the purpose of distinguishing between smooth bifurcation and border collision). Precisely, our definition of structural change as applied to switching converters is as follows.

*A switching converter is said to be structurally changed if its topological sequence in a switching period is altered.*

When no structural changes are involved, power electronics systems may exhibit a variety of smooth bifurcation such as period-doubling bifurcation, Hopf bifurcation, etc., as will be detailed in later chapters. Moreover, it should be apparent that power electronics systems are prone to border collision since operating boundaries exist to separate various operating modes. Two situations are particularly relevant to switching converters, as illustrated in Figure 1.13.

1. *Change of operating mode:* In any switching converter, a boundary exists between continuous and discontinuous conduction modes of operation. Due to the difference in the topological sequence assumed by the converter for the two conduction modes, the converter undergoes a structural change when its operation changes from one mode to another. Crossing the boundary of the two conduction modes would cause a border collision.

2. *Saturating nonlinearity:* Saturating boundaries naturally exist due to the inherent limitation of the range of some control parameters. At such saturating boundaries, the topological sequence is significantly altered. For example, in the voltage-mode buck converter shown in Figure 1.2, the control signal is supposed to hit the ramp signal once per switching period. If this fails to happen due to an excessively wide swing of the control signal, the topological sequence is altered significantly. A border collision thus occurs.

In the past two decades, a few important basic findings regarding bifurcation in switching converters have been established. Some surveys of published work have been conducted by Hamill [58], Hamill, Banerjee and Verghese [59], Nagy [102], Tse [145], and Tse and di Bernardo [148]. Here, we give a brief summary.

1. Voltage-mode controlled buck converters typically undergo period-doubling bifurcations [27, 48, 60], whereas boost converters are more likely to exhibit Hopf bifurcation [5, 68].

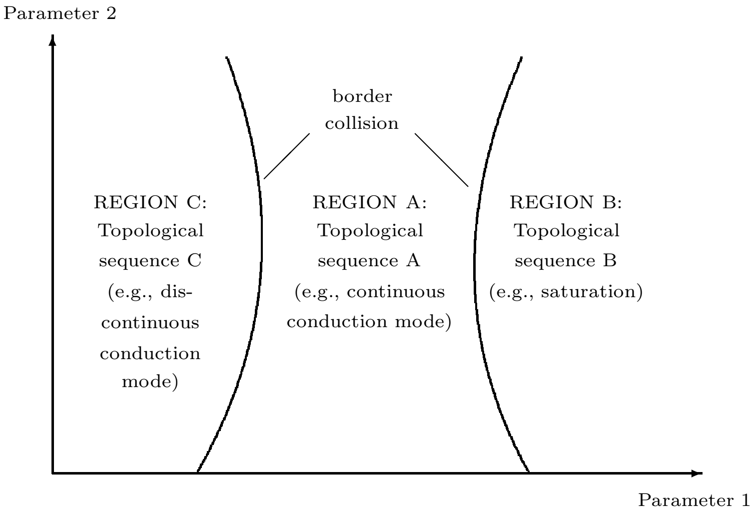



**FIGURE 1.13**
Operating boundaries on parameter space separating regions with different topological sequences. Border collision occurs at the boundaries where the converter experiences a structural change as its topological sequence is altered.

2. Period-doubling is common in buck or boost-type converters operating in discontinuous conduction mode [141, 142] and current-mode controlled converters [25, 38, 150].

3. A variety of bifurcations are possible when other nonlinear control methods are used, e.g., crisis, saddle-node bifurcation, switching-time bifurcation, etc. [45, 59, 73, 94].

4. Border collision is often present to organize the overall bifurcation pattern [8, 10, 14, 172].

In the rest of this book, we will take a detailed look at the bifurcation phenomena that govern the complex behavior of switching power converters. We will begin in Chapter 2 with some important computer and laboratory tools for studying the dynamics of nonlinear systems, and in Chapter 3 we will proceed with the essential modeling techniques for facilitating nonlinear analysis of switching power converters. From Chapter 4, through the end, we will examine some selected power converters, with emphasis on bifurcation phenomena. In the process of studying complex behavior of the various converters, we try to illustrate the investigational approach that we have found effective in dealing with complex behavior in switching power converters.