Attia, John Okyere. "AC Analysis and Network Functions."
Electronics and Circuit Analysis using MATLAB.
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CHAPTER SIX

AC ANALYSIS AND NETWORK FUNCTIONS

This chapter discusses sinusoidal steady state power calculations. Numerical integration is used to obtain the rms value, average power and quadrature power. Three-phase circuits are analyzed by converting the circuits into the frequency domain and by using the Kirchoff voltage and current laws. The unknown voltages and currents are solved using matrix techniques.

Given a network function or transfer function, MATLAB has functions that can be used to (i) obtain the poles and zeros, (ii) perform partial fraction expansion, and (iii) evaluate the transfer function at specific frequencies. Furthermore, the frequency response of networks can be obtained using a MATLAB function. These features of MATLAB are applied in this chapter.

6.1   STEADY STATE AC POWER

Figure 6.1 shows an impedance with voltage across it given by $v(t)$ and current through it $i(t)$.
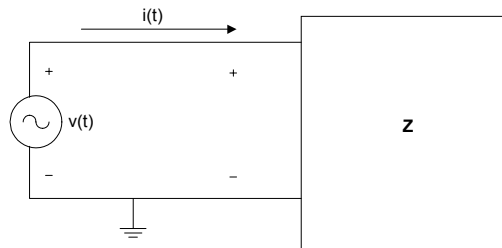

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Figure 6.1 One-Port Network with Impedance Z

The instantaneous power $p(t)$ is

$$p(t) = v(t)i(t) \qquad (6.1)$$

If $v(t)$ and $i(t)$ are periodic with period T, the rms or effective values of the voltage and current are

$$V_{rms} = \sqrt{\frac{1}{T} \int_0^T v^2(t)dt} \qquad (6.2)$$

$$I_{rms} = \sqrt{\frac{1}{T} \int_0^T i^2(t)dt} \qquad (6.3)$$

where

V_{rms} is the rms value of $v(t)$

I_{rms} is the rms value of $i(t)$

The average power dissipated by the one-port network is

$$P = \frac{1}{T} \int_0^T v(t)i(t)dt \qquad (6.4)$$

The power factor, pf, is given as

$$pf = \frac{P}{V_{rms} I_{rms}} \qquad (6.5)$$

For the special case, where both the current $i(t)$ and voltage $v(t)$ are both sinusoidal, that is,

$$v(t) = V_m \cos(wt + \theta_V) \qquad (6.6)$$

and

$$i(t) = I_m \cos(wt + \theta_I) \qquad (6.7)$$

the rms value of the voltage $v(t)$ is

$$V_{rms} = \frac{V_m}{\sqrt{2}} \qquad (6.8)$$

and that of the current is

$$I_{rms} = \frac{I_m}{\sqrt{2}} \qquad (6.9)$$

The average power P is

$$P = V_{rms}I_{rms}\cos(\theta_V - \theta_I) \qquad (6.10)$$

The power factor, pf, is

$$pf = \cos(\theta_V - \theta_I) \qquad (6.11)$$

The reactive power Q is

$$Q = V_{rms}I_{rms}\sin(\theta_V - \theta_I) \qquad (6.12)$$

and the complex power, S, is

$$S = P + jQ \qquad (6.13)$$

$$S = V_{rms}I_{rms}\left[\cos(\theta_V - \theta_I) + j\sin(\theta_V - \theta_I)\right] \qquad (6.14)$$

Equations (6.2) to (6.4) involve the use of integration in the determination of the rms value and the average power. MATLAB has two functions, quad and quad8, for performing numerical function integration.

6.1.1 MATLAB Functions quad and quad8

The **quad** function uses an adaptive, recursive Simpson's rule. The **quad8** function uses an adaptive, recursive Newton Cutes 8 panel rule. The quad8 function is better than the quad at handling functions with "soft" singularities such as $\int \sqrt{x}dx$. Suppose we want to find q given as

$$q = \int_a^b funct(x)dx$$

The general forms of quad and quad8 functions that can be used to find q are

$$quad('funct', a, b, tol, trace)$$

$$quad8('funct', a, b, tol, trace)$$

where

funct	is a MATLAB function name (in quotes) that returns a vector of values of $f(x)$ for a given vector of input values x.
a	is the lower limit of integration.
b	is the upper limit of integration.
tol	is the tolerance limit set for stopping the iteration of the numerical integration. The iteration continues until the relative error is less than tol. The default value is 1.0e-3.
trace	allows the plot of a graph showing the process of the numerical integration. If the trace is nonzero, a graph is plotted. The default value is zero.

Example 6.1 shows the use of the quad function to perform alternating current power calculations.

Example 6.1

For Figure 6.1, if $v(t) = 10\cos(120\pi t + 30^0)$ and $i(t) = 6\cos(120\pi t + 60^0)$. Determine the average power, rms value of $v(t)$ and the power factor using (a) analytical solution and (b) numerical solution.

Solution

MATLAB Script

```
diary  ex6_1.dat
% This program computes the average power, rms value and
% power factor using quad function. The analytical and
% numerical results are compared.
% numerical calculations
```

```
T = 2*pi/(120*pi); % period of the sin wave
a = 0; % lower limit of integration
b = T; % upper limit of integration
x = 0:0.02:1;
t = x.*b;
v_int = quad('voltage1', a, b);
v_rms = sqrt(v_int/b);  % rms of voltage
i_int = quad('current1',a,b);
i_rms = sqrt(i_int/b);  % rms of current

p_int = quad('inst_pr', a, b);
p_ave = p_int/b;    % average power
pf = p_ave/(i_rms*v_rms); % power factor
%
% analytical solution
%
p_ave_an = (60/2)*cos(30*pi/180);  % average power
v_rms_an = 10.0/sqrt(2);
pf_an = cos(30*pi/180);

% results are printed
fprintf('Average power, analytical %f \n Average power, numerical:
%f \n', p_ave_an,p_ave)
fprintf('rms voltage, analytical: %f \n rms voltage, numerical: %f \n',
v_rms_an, v_rms)
fprintf('power factor, analytical: %f \n power factor, numerical: %f \n',
pf_an, pf)
diary
```

The following functions are used in the above m-file:

```
function vsq = voltage1(t)
% voltage1  This function is used to
%       define the voltage function
vsq = (10*cos(120*pi*t + 60*pi/180)).^2;
end

function isq = current1(t)
% current1  This function is to define the current
%
isq = (6*cos(120*pi*t + 30.0*pi/180)).^2;
end
```

```
function pt = inst_pr(t)
% inst_pr  This function is used to define
%         instantaneous power obtained by multiplying
%         sinusoidal voltage and current
it = 6*cos(120*pi*t + 30.0*pi/180);
vt = 10*cos(120*pi*t + 60*pi/180);
pt = it.*vt;
end
```

The results obtained are

Average power, analytical 25.980762
Average power, numerical: 25.980762
rms voltage, analytical: 7.071068
rms voltage, numerical: 7.071076
power factor, analytical: 0.866025
power factor, numerical: 0.866023

From the results, it can be seen that the two techniques give almost the same answers.

6.2   SINGLE- AND THREE-PHASE AC CIRCUITS

Voltages and currents of a network can be obtained in the time domain. This normally involves solving differential equations. By transforming the differential equations into algebraic equations using phasors or complex frequency representation, the analysis can be simplified. For a voltage given by

$$v(t) = V_m e^{\sigma t} \cos(wt + \theta)$$

$$v(t) = \text{Re}\left[V_m e^{\sigma t} \cos(wt + \theta)\right] \tag{6.15}$$

the phasor is

$$V = V_m e^{j\theta} = V_m \angle \theta \tag{6.16}$$

and the complex frequency s is

$$s = \sigma + jw \tag{6.17}$$

When the voltage is purely sinusoidal, that is

$$v_2(t) = V_{m2} \cos(wt + \theta_2) \tag{6.18}$$

then the phasor

$$V_2 = V_{m2} e^{j\theta_2} = V_{m2} \angle \theta_2 \tag{6.19}$$

and complex frequency is purely imaginary, that is,

$$s = jw \tag{6.20}$$

To analyze circuits with sinusoidal excitations, we convert the circuits into the s-domain with $s = jw$. Network analysis laws, theorems, and rules are used to solve for unknown currents and voltages in the frequency domain. The solution is then converted into the time domain using inverse phasor transformation. For example, Figure 6.2 shows an RLC circuit in both the time and frequency domains.
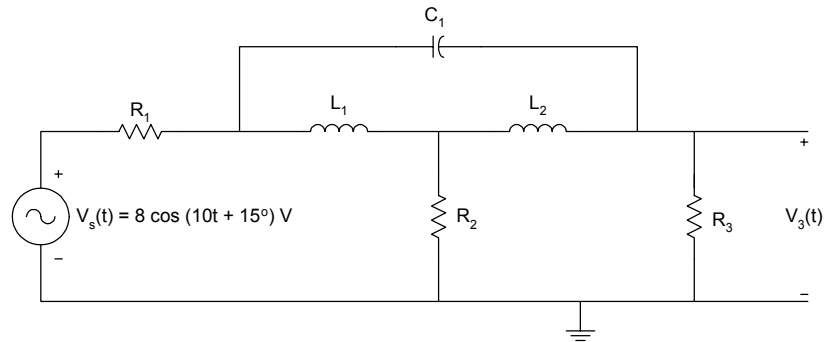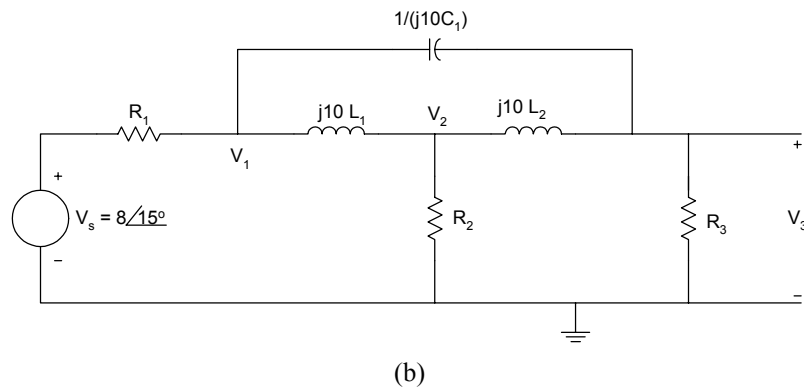

(a)

Figure 6.2 RLC Circuit with Sinusoidal Excitation (a) Time
Domain (b) Frequency Domain Equivalent

If the values of R_1, R_2, R_3, L_1, L_2 and C_1 are known, the voltage V_3 can be obtained using circuit analysis tools. Suppose V_3 is

$$V_3 = V_{m3} \angle \theta_3,$$

then the time domain voltage V_3 *(t)* is

$$v_3(t) = V_{m3} \cos(wt + \theta_3)$$

The following two examples illustrate the use of MATLAB for solving one-phase circuits.

Example 6.2

In Figure 6.2, if R_1 = 20 Ω, R_2 = 100Ω, R_3 = 50 Ω, and L_1 = 4 H, L_2 = 8 H and C_1 = 250µF, find $v_3(t)$ when $w = 10$ rad/s.

Solution

Using nodal analysis, we obtain the following equations.

At node 1,

$$\frac{V_1 - V_s}{R_1} + \frac{V_1 - V_2}{j10L_1} + \frac{V_1 - V_3}{1/(j10C_1)} = 0 \tag{6.21}$$

At node 2,

$$\frac{V_2 - V_1}{j10L_1} + \frac{V_2}{R_2} + \frac{V_2 - V_3}{j10L_2} = 0 \tag{6.22}$$

At node 3,

$$\frac{V_3}{R_3} + \frac{V_3 - V_2}{j10L_2} + \frac{V_3 - V_1}{1/(j10C_1)} = 0 \tag{6.23}$$

Substituting the element values in the above three equations and simplifying, we get the matrix equation

$$\begin{bmatrix} 0.05 - j0.0225 & j0.025 & -j0.0025 \\ j0.025 & 0.01 - j0.0375 & j0.0125 \\ -j0.0025 & j0.0125 & 0.02 - j0.01 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0.4\angle 15^0 \\ 0 \\ 0 \end{bmatrix}$$

The above matrix can be written as

$$[Y][V] = [I].$$

We can compute the vector [v] using the MATLAB command

$$V = inv(Y) * I$$

where

$inv(Y)$ is the inverse of the matrix $[Y]$.

A MATLAB program for solving V_3 is as follows:

MATLAB Script

```
diary ex6_2.dat
% This program computes the nodal voltage v3 of circuit Figure 6.2
```

```
% Y is the admittance matrix; % I is the current matrix
% V is the voltage vector

Y = [0.05-0.0225*j   0.025*j      -0.0025*j;
     0.025*j       0.01-0.0375*j  0.0125*j;
    -0.0025*j       0.0125*j       0.02-0.01*j];

c1 = 0.4*exp(pi*15*j/180);
I = [c1
     0
     0];  % current vector entered as column vector

V = inv(Y)*I;  % solve for nodal voltages
v3_abs = abs(V(3));
v3_ang = angle(V(3))*180/pi;

fprintf('voltage V3, magnitude: %f \n voltage V3, angle in degree:
%f', v3_abs, v3_ang)
diary
```

The following results are obtained:

> voltage V3, magnitude: 1.850409
> voltage V3, angle in degree: -72.453299

From the MATLAB results, the time domain voltage $v_3(t)$ is

$$v_3(t) = 1.85\cos(10t - 72.45^0) \text{ V}$$

Example 6.3

For the circuit shown in Figure 6.3, find the current $i_1(t)$ and the voltage $v_C(t)$.
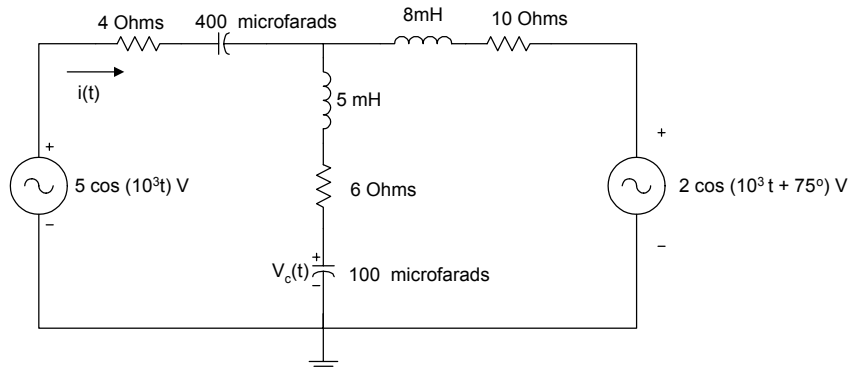
Figure 6.3 Circuit with Two Sources

Solution

Figure 6.3 is transformed into the frequency domain. The resulting circuit is shown in Figure 6.4. The impedances are in ohms.
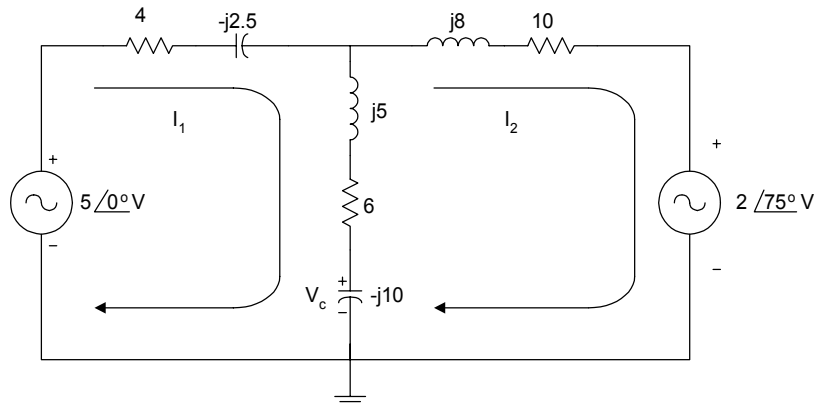

Figure 6.4 Frequency Domain Equivalent of Figure 6.3

Using loop analysis, we have

$$-5\angle 0^0 + (4 - j2.5)I_1 + (6 + j5 - j10)(I_1 - I_2) = 0 \quad (6.24)$$

$$(10 + j8)I_2 + 2\angle 75^0 + (6 + j5 - j10)(I_2 - I_1) = 0 \quad (6.25)$$

Simplifying, we have

$$(10 - j7.5)I_1 - (6 - j5)I_2 = 5\angle 0^0$$

$$-(6 - j5)I_1 + (16 + j3)I_2 = -2\angle 75^0$$

In matrix form, we obtain

$$\begin{bmatrix} 10 - j7.5 & -6 + j5 \\ -6 + j5 & 16 + j3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 5\angle 0^0 \\ -2\angle 75^0 \end{bmatrix}$$

The above matrix equation can be rewritten as

$$[Z][I] = [V].$$

We obtain the current vector $[I]$ using the MATLAB command

$$I = inv(Z) * V$$

where $inv(Z)$ is the inverse of the matrix $[Z]$.

The voltage V_C can be obtained as

$$V_C = (-j10)(I_1 - I_2)$$

A MATLAB program for determining I_1 and V_a is as follows:

MATLAB Script

```
diary ex6_3.dat
% This programs calculates the phasor current I1 and
% phasor voltage Va.
% Z is impedance matrix
% V is voltage vector
% I is current vector

Z = [10-7.5*j  -6+5*j;
     -6+5*j   16+3*j];

b = -2*exp(j*pi*75/180);
```

```
V = [5
      b];  % voltage vector in column form

I = inv(Z)*V; % solve for loop currents
i1 = I(1);
i2 = I(2);

Vc = -10*j*(i1 - i2);
i1_abs = abs(I(1));
i1_ang = angle(I(1))*180/pi;
Vc_abs = abs(Vc);
Vc_ang = angle(Vc)*180/pi;

%results are printed
fprintf('phasor current i1, magnitude: %f \n phasor current i1, angle in
degree: %f \n', i1_abs,i1_ang)
fprintf('phasor voltage Vc, magnitude: %f \n phasor voltage Vc, angle
in degree: %f \n',Vc_abs,Vc_ang)
diary
```

The following results were obtained:

```
phasor current i1, magnitude: 0.387710
phasor current i1, angle in degree: 15.019255
phasor voltage Vc, magnitude: 4.218263
phasor voltage Vc, angle in degree: -40.861691
```

The current $i_1(t)$ is

$$i_1(t) = 0.388\cos(10^3 t + 15.02^0) \quad \text{A}$$

and the voltage $v_C(t)$ is

$$v_C(t) = 4.21\cos(10^3 t - 40.86^0) \quad \text{V}$$

Power utility companies use three-phase circuits for the generation, transmission and distribution of large blocks of electrical power. The basic structure of a three-phase system consists of a three-phase voltage source connected to a three-phase load through transformers and transmission lines. The three-phase voltage source can be wye- or delta-connected. Also the three-phase load can be delta- or wye-connected. Figure 6.5 shows a 3-phase system with wye-connected source and wye-connected load.

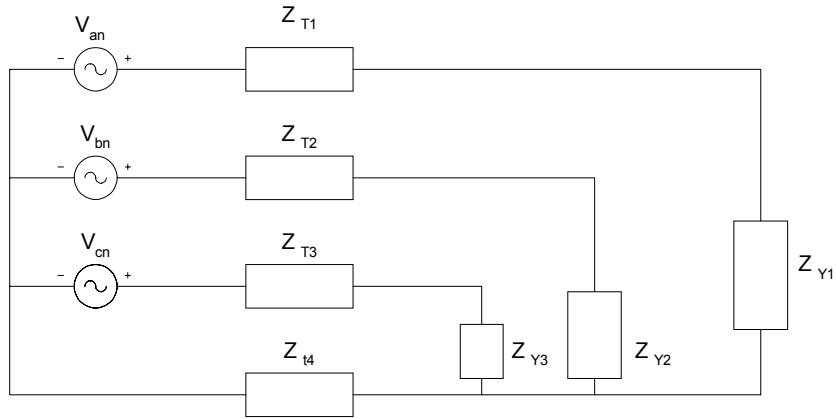**Figure 6.5** 3-phase System, Wye-connected Source and Wye-connected Load
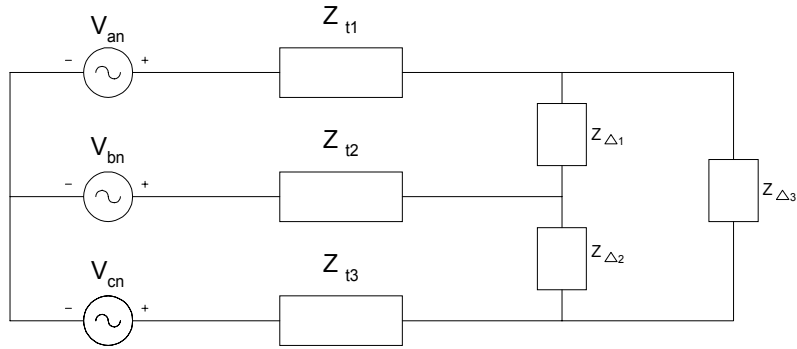

Figure 6.6 3-phase System, Wye-connected Source and Delta-connected Load

For a balanced abc system, the voltages V_{an}, V_{bn}, V_{cn} have the same magnitude and they are out of phase by 120^0. Specifically, for a balanced abc system, we have

$$V_{an} = V_P \angle 0^0$$
$$V_{bn} = V_P \angle -120^0 \qquad (6.26)$$
$$V_{cn} = V_P \angle 120^0$$

For cba system

$$V_{an} = V_P \angle 0^0$$
$$V_{bn} = V_P \angle 120^0 \qquad (6.27)$$
$$V_{cn} = V_P \angle -120^0$$

The wye-connected load is balanced if

$$Z_{Y1} = Z_{Y2} = Z_{Y3} \qquad (6.28)$$

Similarly, the delta-connected load is balanced if

$$Z_{\Delta 1} = Z_{\Delta 2} = Z_{\Delta 3} \qquad (6.29)$$

We have a balanced three-phase system of Equations (6.26) to (6.29) that are satisfied with the additional condition

$$Z_{T1} = Z_{T2} = Z_{T3} \qquad (6.30)$$

Analysis of balanced three-phase systems can easily be done by converting the three-phase system into an equivalent one-phase system and performing simple hand calculations. The method of symmetrical components can be used to analyze unbalanced three-phase systems. Another method that can be used to analyze three-phase systems is to use KVL and KCL. The unknown voltage or currents are solved using MATLAB. This is illustrated by the following example.

Example 6.4

In Figure 6.7, showing an unbalanced wye-wye system, find the phase voltages V_{AN}, V_{BN} and V_{CN}.

Solution

Using KVL, we can solve for I_1, I_2, I_3. From the figure, we have

$$110\angle 0^0 = (1 + j1)I_1 + (5 + j12)I_1 \qquad (6.31)$$

$$110\angle-120^0 = (1-j2)I_2 + (3+j4)I_2 \qquad (6.32)$$

$$110\angle120^0 = (1-j0.5)I_3 + (5-j12)I_3 \qquad (6.33)$$

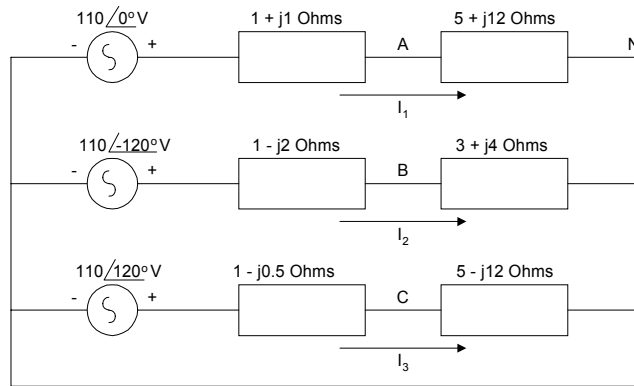**Figure 6.7**   Unbalanced Three-phase System

Simplifying Equations (6.31), (6.32) and (6.33), we have

$$110\angle0^0 = (6+j13)I_1 \qquad (6.34)$$

$$110\angle-120^0 = (4+j2)I_2 \qquad (6.35)$$

$$110\angle120^0 = (6-j12.5)I_3 \qquad (6.36)$$

and expressing the above three equations in matrix form, we have

$$\begin{bmatrix} 6+j13 & 0 & 0 \\ 0 & 4+j2 & 0 \\ 0 & 0 & 6-j12.5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 110\angle0^0 \\ 110\angle-120^0 \\ 110\angle120^0 \end{bmatrix}$$

The above matrix can be written as

$$[Z][I] = [V]$$

We obtain the vector $[I]$ using the MATLAB command

$$I = inv(Z) * V$$

The phase voltages can be obtained as

$$V_{AN} = (5 + j12)I_1$$

$$V_{BN} = (3 + j4)I_2$$

$$V_{CN} = (5 - j12)(I_3)$$

The MATLAB program for obtaining the phase voltages is

MATLAB Script

```
diary ex6_4.dat
% This program calculates the phasor voltage of an
% unbalanced three-phase system
% Z is impedance matrix
% V is voltage vector and
% I is current vector
Z = [6-13*j   0      0;
     0       4+2*j   0;
     0        0      6-12.5*j];
c2 = 110*exp(j*pi*(-120/180));
c3 = 110*exp(j*pi*(120/180));

V = [110; c2; c3]; % column voltage vector
I = inv(Z)*V;  % solve for loop currents
% calculate the phase voltages
%
Van = (5+12*j)*I(1);
Vbn = (3+4*j)*I(2);
Vcn = (5-12*j)*I(3);
Van_abs = abs(Van);
Van_ang = angle(Van)*180/pi;
Vbn_abs = abs(Vbn);
Vbn_ang = angle(Vbn)*180/pi;
Vcn_abs = abs(Vcn);
Vcn_ang = angle(Vcn)*180/pi;

% print out results
```

```
fprintf('phasor voltage Van,magnitude: %f \n phasor voltage Van, an-
gle in degree: %f \n', Van_abs, Van_ang)
fprintf('phasor voltage Vbn,magnitude: %f \n phasor voltage Vbn, an-
gle in degree: %f \n', Vbn_abs, Vbn_ang)
fprintf('phasor voltage Vcn,magnitude: %f \n phasor voltage Vcn, an-
gle in degree: %f \n', Vcn_abs, Vcn_ang)
diary
```

The following results were obtained:

```
phasor voltage Van,magnitude: 99.875532
phasor voltage Van, angle in degree: 132.604994
phasor voltage Vbn,magnitude: 122.983739
phasor voltage Vbn, angle in degree: -93.434949
phasor voltage Vcn,magnitude: 103.134238
phasor voltage Vcn, angle in degree: 116.978859
```

6.3   NETWORK CHARACTERISTICS

Figure 6.8 shows a linear network with input $x(t)$ and output $y(t)$. Its complex frequency representation is also shown.
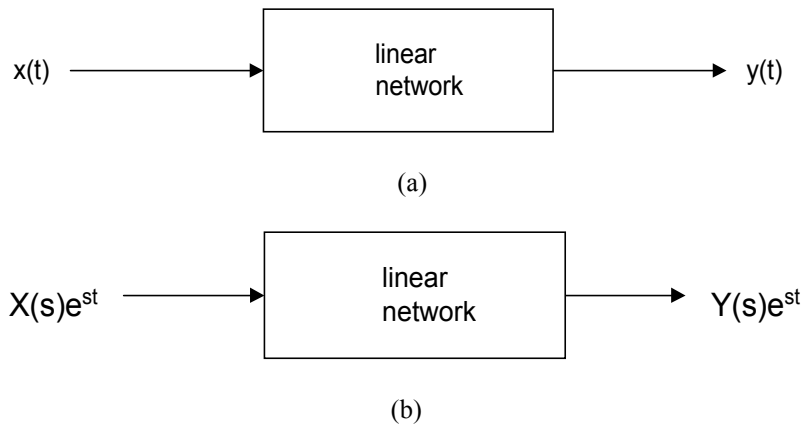

(a)

(b)

Figure 6.8 Linear Network Representation (a) Time Domain
(b) s- domain

In general, the input $x(t)$ and output $y(t)$ are related by the differential equation

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

(6.37)

where a_n, a_{n-1}, ..., a_0, b_m, b_{m-1}, ... b_0 are real constants.

If $x(t) = X(s)e^{st}$, then the output must have the form $y(t) = Y(s)e^{st}$, where $X(s)$ and $Y(s)$ are phasor representations of $x(t)$ and $y(t)$. From equation (6.37), we have

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y(s) e^{st} =$$

$$(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) X(s) e^{st}$$

(6.38)

and the network function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0}$$

(6.39)

The network function can be rewritten in factored form

$$H(s) = \frac{k(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

(6.40)

where

k is a constant

$z_1, z_2, \ldots, z_m$ are zeros of the network function.

$p_1, p_2, \ldots, p_n$ are poles of the network function.

The network function can also be expanded using partial fractions as

$$H(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \quad \ldots \quad + \frac{r_n}{s - p_n} + k(s)$$

(6.41)

6.3.1   MATLAB functions roots, residue and polyval

MATLAB has the function **roots** that can be used to obtain the poles and zeros of a network function. The MATLAB function **residue** can be used for partial fraction expansion.   Furthermore, the MATLAB function **polyval** can be used to evaluate the network function.

The MATLAB function **roots** determines the roots of a polynomial. The general form of the roots function is

$$r = roots(p) \tag{6.42}$$

where
> p is a vector containing the coefficients of the polynomial in
>> descending order
>
> r is a column vector containing the roots of the polynomials

For example, given the polynomial

$$f(x) = x^3 + 9x^2 + 23x + 15$$

the commands to compute and print out the roots of $f(x)$ are

    p = [1 9 23 15]
    r = roots (p)

and the values printed are

    r =
        -1.0000
        -3.0000
        -5.0000

Given the roots of a polynomial, we can obtain the coefficients of the polynomial by using the MATLAB function **poly**

Thus

$$S = poly ([-1 \quad -3 \quad -5]^1) \tag{6.43}$$

will give a row vector s given as

S =
    1.0000   9.0000   23.0000   15.0000

The coefficients of S are the same as those of p.

The MATLAB function **polyval** is used for polynomial evaluation. The general form of polyval is

$$polyval(p, x) \tag{6.44}$$

where

  p is a vector whose elements are the coefficients of a polynomial in
     descending powers
  $polyval(p, x)$ is the value of the polynomial evaluated at x

For example, to evaluate the polynomial

$$f(x) = x^3 - 3x^2 - 4x + 15$$

at $x = 2$, we use the command

    p = [1 -3 -4   15];
    polyval(p, 2)

Then we get

    ans =
        3

The MATLAB function **residue** can be used to perform partial fraction expansion. Assuming $H(s)$ is the network function, since $H(s)$ may represent an improper fraction, we may express $H(s)$ as a mixed fraction

$$H(s) = \frac{B(s)}{A(s)} \tag{6.45}$$

$$H(s) = \sum_{n=0}^{N} k_n s^n + \frac{N(s)}{D(s)} \qquad (6.46)$$

where

$$\frac{N(s)}{D(s)} \text{ is a proper fraction}$$

From equations (6.41) and (6.46), we get

$$H(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \quad .... \quad + \frac{r_n}{s - p_n} + \sum_{n=0}^{N} k_n s^n \qquad (6.47)$$

Given the coefficients of the numerator and denominator polynomials, the MATLAB **residue** function provides the values of $r_1, r_2, ... r_n , p_1, p_2, ...p_n$, an d $k_1, k_2, ...k_n$. The general form of the residue function is

$$[r, p, k] = residue(num, den) \qquad (6.48)$$

where

    num   is a row vector whose entries are the coefficients of the numerator polynomial in descending order

    den   is a row vector whose entries are the coefficient of the denominator polynomial in descending order

    r   is returned as a column vector

    p   (pole locations) is returned as a column vector

    k   (direct term) is returned as a row vector

The command

$$[num, den] = residue(r, p, k) \qquad (6.49)$$

Converts the partial fraction expansion back to the polynomial ratio

$$H(s) = \frac{B(s)}{A(s)}$$

For example, given

$$H(s) = \frac{4s^4 + 3s^3 + 6s^2 + 10s + 20}{s^4 + 2s^3 + 5s^2 + 2s + 8} \tag{6.50}$$

for the above network function, the following commands will perform partial fraction expansion

        num = [4 3 6 10 20];
        den = [1 2 5 2 8];
        [r, p, k] = residue(num, den)   (6.51)

and we shall get the following results

        r =
            -1.6970 + 3.0171i
            -1.6970 - 3.0171i
            -0.8030 - 0.9906i
            -0.8030 + 0.9906i

        p =
            -1.2629 + 1.7284i
            -1.2629 - 1.7284i
             0.2629 + 1.2949i
             0.2629 - 1.2949i

        k =
             4

The following two examples show how to use MATLAB function **roots** to find poles and zeros of circuits.

Example 6.5

For the circuit shown below, (a) Find the network function $H(s) = \dfrac{V_o(s)}{V_S(s)}$

(b) Find the poles and zeros of $H(s)$, and

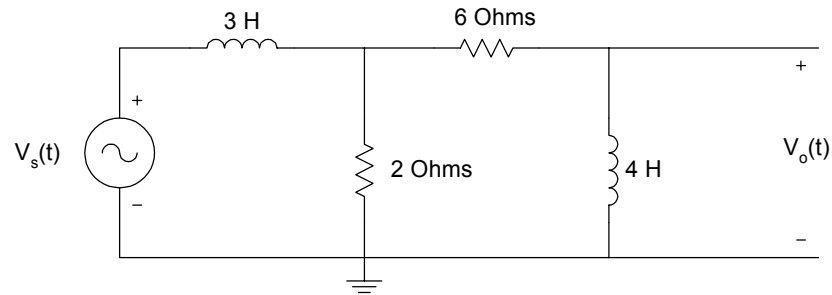(c) if $v_S(t) = 10e^{-3t} \cos(2t + 40^0)$, find $v_0(t)$.

Figure 6.9 Circuit for Example 6.5
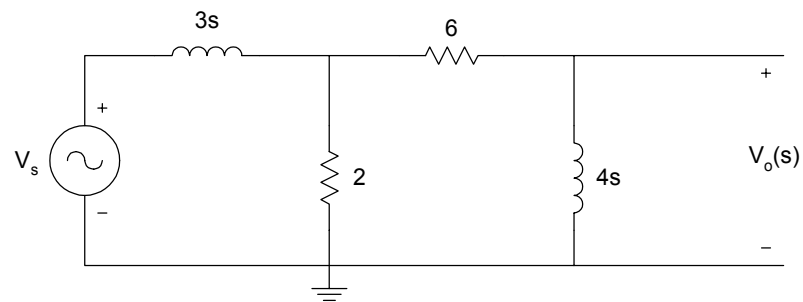
Solution

In the s-domain, the above figure becomes

Figure 6.10 S-domain Equivalent Circuit of Figure 6.9

$$\frac{V_0(s)}{V_S(s)} = \frac{V_0(s)}{V_X(s)} \frac{V_X(s)}{V_S(s)} = \frac{4s}{(6+4s)} \frac{[2\|(6+4s)]}{[(2\|(6+4s))+3s]}$$

Simplifying, we get

$$\frac{V_0(s)}{V_S(s)} = \frac{4s^2 + 6s}{6s^3 + 25s^2 + 30s + 9} \tag{6.52}$$

The phasor voltage $V_S = 10\angle 40^o$; $\quad s = -3 + j2$

$$V_0(s) = (10\angle 40^o)H(s)\Big|_{s=-3+j2}$$

(b, c) MATLAB is used to find the poles, zeros and $v_0(t)$.

MATLAB Script

```
diary ex6_5.dat
% Program for poles and zeros
num = [4  6  0];
den = [6  25  30  9];
disp('the zeros are')
z = roots(num)
disp('the poles are')
p = roots(den)
% program to evaluate transfer function and
% find the output voltage
s1 = -3+2*j;
n1 = polyval(num,s1);
d1 = polyval(den,s1);
vo = 10.0*exp(j*pi*(40/180))*n1/d1;
vo_abs = abs(vo);
vo_ang = angle(vo)*180/pi;
% print magnitude and phase of output voltage
fprintf('phasor voltage vo, magnitude: %f \n phasor voltage vo, angle
in degrees: %f', vo_abs, vo_ang)
diary
```

MATLAB results are

Zeros

```
z =
     0
   -1.5000
```

Poles

```
p =
   -2.2153
   -1.5000
   -0.4514
```

phasor voltage vo, magnitude: 3.453492

phasor voltage vo, angle in degrees: -66.990823

From the results, the output voltage is given as

$$v(t) = 3.45e^{-3t} \cos(2t - 66.99^0)$$

Example 6.6

Find the inverse Laplace transform of

$$G(s) = \frac{10s^2 + 20s + 40}{s^3 + 12s^2 + 47s + 60}$$

Solution

MATLAB Script

```
diary ex6_6.dat
% MATLAB is used to do the partial fraction expansion
%
num = [10 20 40];
den = [1 12 47 60];

% we get the following results
[r, p, k] = residue(num,den)
diary
```

MATLAB results are

```
r =
      95.0000
    -120.0000
      35.0000

p =
     -5.0000
     -4.0000
     -3.0000

k =
```

[]

From the results, we get

$$G(s) = \frac{95}{s+5} - \frac{120}{s+4} + \frac{35}{s+3}$$

and the inverse Laplace transform is

$$g(t) = 35e^{-3t} - 120e^{-4t} + 95e^{-5t} \tag{6.53}$$

6.4   FREQUENCY RESPONSE

The general form of a transfer function of an analog circuit is given in Equation (6.39). It is repeated here.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0}$$

More specifically, for a second-order analog filter, the following transfer functions can be obtained:

(i) Lowpass

$$H_{LP}(s) = \frac{k_1}{s^2 + Bs + w_0^2} \tag{6.54}$$

(ii) Highpass

$$H_{HP}(s) = \frac{k_2 s^2}{s^2 + Bs + w_0^2} \tag{6.55}$$

(iii) Bandpass
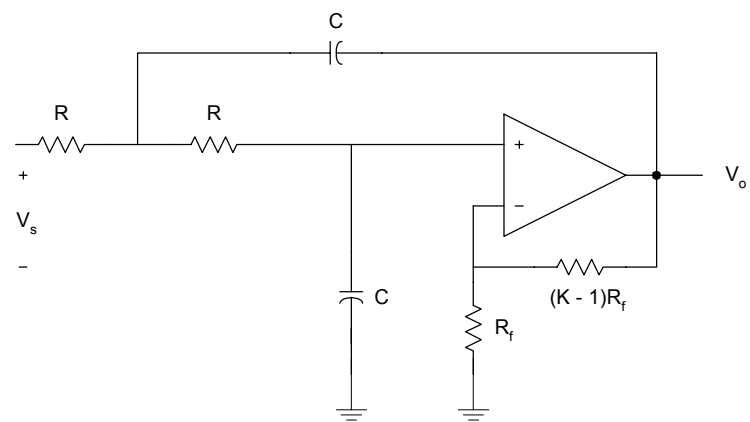
$$H_{BP}(s) = \frac{k_3 s}{s^2 + Bs + w_0^2} \tag{6.56}$$

(iv) Bandreject

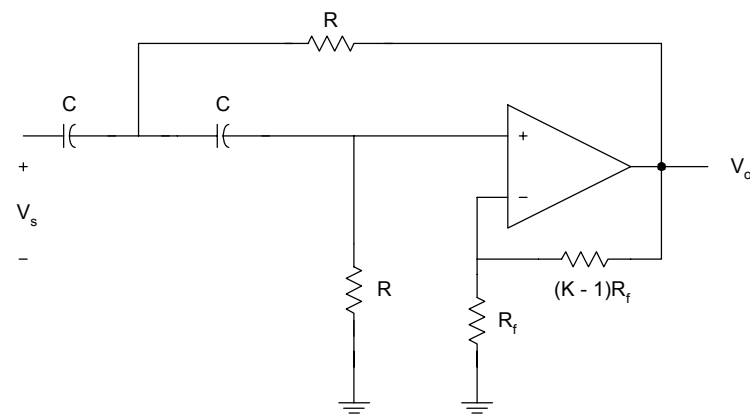$$H_{BR}(s) = \frac{k_4 s^2 + k_5}{s^2 + Bs + w_0^2} \qquad (6.57)$$

where

$$k_1, k_2, k_3, k_4, B \text{ and } w_0 \text{ are constants}$$

Figure 6.11 shows the circuit diagram of some filter sections.
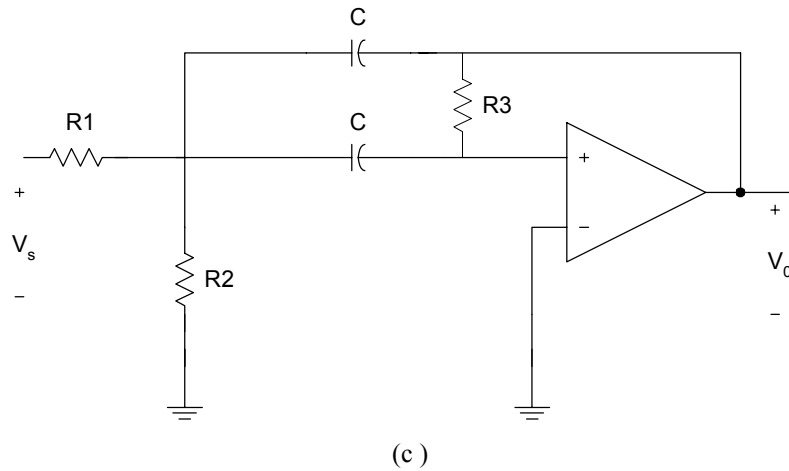

(a)

(b)

(c)

Figure 6.11 Active Filters (a) Lowpass, (b) Highpass and
(c) Bandpass

Frequency response is the response of a network to sinusoidal input signal. If we substitute $s = jw$ in the general network function, $H(s)$, we get

$$H(s)\Big|_{s=jw} = M(w)\angle\theta(w) \tag{6.58}$$

where
$$M(w) = \left|H(jw)\right| \tag{6.59}$$

and

$$\theta(w) = \angle H(jw) \tag{6.60}$$

The plot of M (ω) versus ω is the magnitude characteristics or response. Also, the plot of $\theta(w)$ versus ω is the phase response. The magnitude and phase characteristics can be obtained using MATLAB function **freqs**.

6.4.1 MATLAB function freqs

MATLAB function **freqs** is used to obtain the frequency response of transfer function $H(s)$. The general form of the frequency function is

$$hs = freqs(num, den, range) \tag{6.61}$$

where

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_1 s + a_0} \tag{6.62}$$

$$num = \begin{bmatrix} b_m & b_{\cdot m-1} & \dots & b_1 & b_0 \end{bmatrix} \tag{6.63}$$

$$den = \begin{bmatrix} a_n & a_{n-1} & \dots & a_1 & a_0 \end{bmatrix} \tag{6.64}$$

$range$ is range of frequencies for case

hs is the frequency response (in complex number form)

Suppose we want to graph the frequency response of the transfer function given as

$$H(s) = \frac{2s^2 + 4}{s^2 + 4s + 16} \tag{6.65}$$

We can use the following commands to find the magnitude characteristics

```
num = [2 0 4];
den = [1 4 16];
w = logspace(-2, 4);
h = freqs(num, den, w);
f = w/(2*pi);
mag = 20*log10(abs(h));
semilogx(f, mag)
title('Magnitude Response')
xlabel('Frequency, Hz')
ylabel('Gain, dB')
```
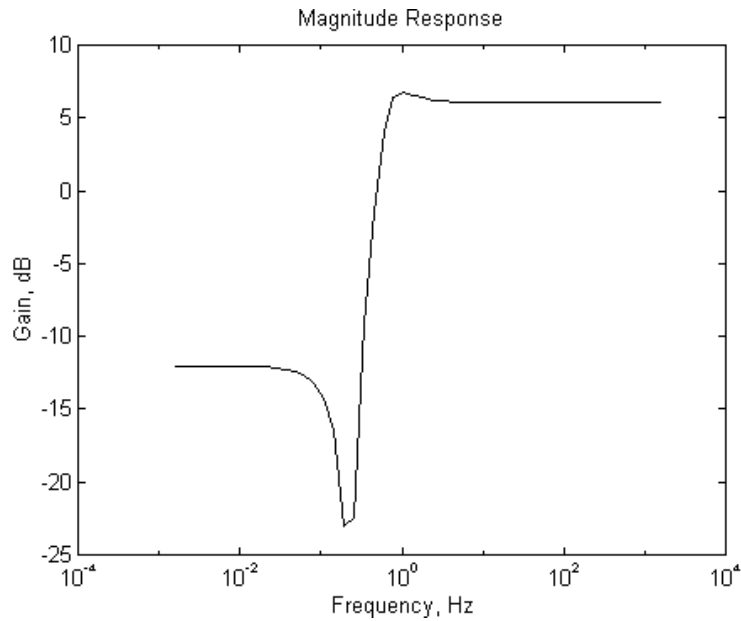
The frequency response is shown in Figure 6.12.

Figure 6.12 Magnitude Response of Equation (6.65)

The following example shows how to obtain and plot the frequency response of an RLC circuit.

Example 6.7

For the RLC circuit shown in Figure 6.13, (a) show that the transfer function is

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{s\dfrac{R}{L}}{s^2 + s\dfrac{R}{L} + \dfrac{1}{LC}} \tag{6.66}$$

(b) If $L = 5$ H, $C = 1.12\ \mu F$, and $R = 10000\ \Omega$, plot the frequency response.

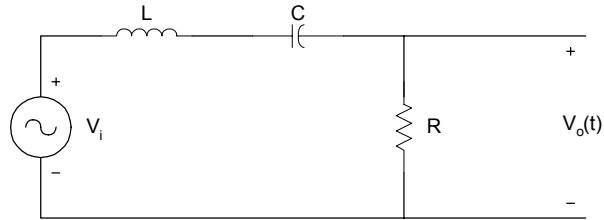(c) What happens when $R = 100\ \Omega$, but L and C remain unchanged?

Figure 6.13 RLC Circuit

Solution

(a) In the frequency domain,

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{R}{R + sL + \dfrac{1}{sC}} = \frac{sCR}{s^2 LC + sCR + 1} \qquad (6.67)$$

which is

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{s\dfrac{R}{L}}{s^2 + s\dfrac{R}{L} + \dfrac{1}{LC}}$$

Parts (b) and (c) are solved using MATLAB.

MATLAB Script

```
% Frequency response of RLC filter
%
l = 5;
c = 1.25e-6;
r1 = 10000;
r2 = 100;

num1 = [r1/l 0];
den1 = [1 r1/l  1/(l*c)];

w = logspace(1,4);
h1 = freqs(num1,den1,w);
```

```
f = w/(2*pi);
mag1 = abs(h1);
phase1 = angle(h1)*180/pi;

num2 = [r2/l 0];
den2 = [1  r2/l  1/(l*c)];
h2 = freqs(num2,den2,w);
mag2 = abs(h2);
phase2 = angle(h2)*180/pi;

% Plot the response

subplot(221), loglog(f, mag1,'.')
title('magnitude response R=10K')
ylabel('magnitude')

subplot(222), loglog(f,mag2,'.')
title('magnitude response R=.1K')
ylabel('magnitude')

subplot(223), semilogx(f, phase1,'.')
title('phase response R=10K'),...
xlabel('Frequency, Hz'), ylabel('angle in degrees')

subplot(224), semilogx(f, phase2,'.')
title('phase response R=.1K'),...
xlabel('Frequency, Hz'), ylabel('angle in degrees')
```

The plots are shown in Figure 6.14. As the resistance is decreased from 10,000 to 100 Ohms, the bandwidth of the frequency response decreases and the quality factor of the circuit increases.
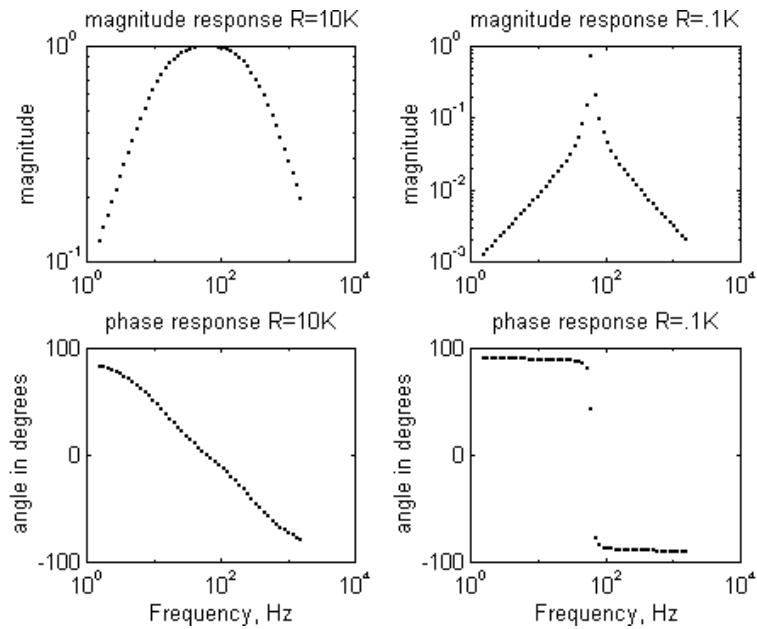
Figure 6.14 Frequency Response of an RLC Circuit

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EXERCISES

6.1   If $v(t)$ is periodic with one period of $v(t)$ given as

$$v(t) = 16(1 - e^{-6t}) \text{ V} \quad 0 \le t < 2 \text{ s}$$

(a) Use MATLAB to find the rms value of $v(t)$

(b) Obtain the rms value of $v(t)$ using analytical technique.
       Compare your result with that obtained in part (a).

(c) Find the power dissipated in the 4-ohm resistor when the
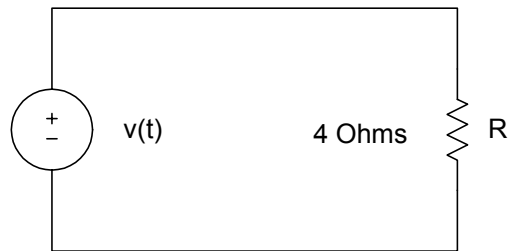   voltage $v(t)$ is applied across the 4-ohm resistor.

Figure P6.1   Resistive Circuit for part (c)

6.2   A balanced Y-Y positive sequence system has phase voltage of the
source $V_{an} = 120\angle 0^0$ rms if the load impedance per phase is
$(11 + j4.5)\,\Omega$, and the transmission line has an impedance per phase
of $(1 + j0.5)\,\Omega$.

(a)   Use analytical techniques to find the magnitude of the line
         current, and the power delivered to the load.

(b)   Use MATLAB to solve for the line current and the power
         delivered to the load.

(c)   Compare the results of parts (a) and (b).

6.3   For the unbalanced 3-phase system shown in Figure P6.3, find the currents I_1, I_2, I_3 and hence I_{bB}. Assume that $Z_A = 10 + j5\,\Omega$, $Z_B = 15 + j7\,\Omega$ and $Z_C = 12 - j3\,\Omega$.
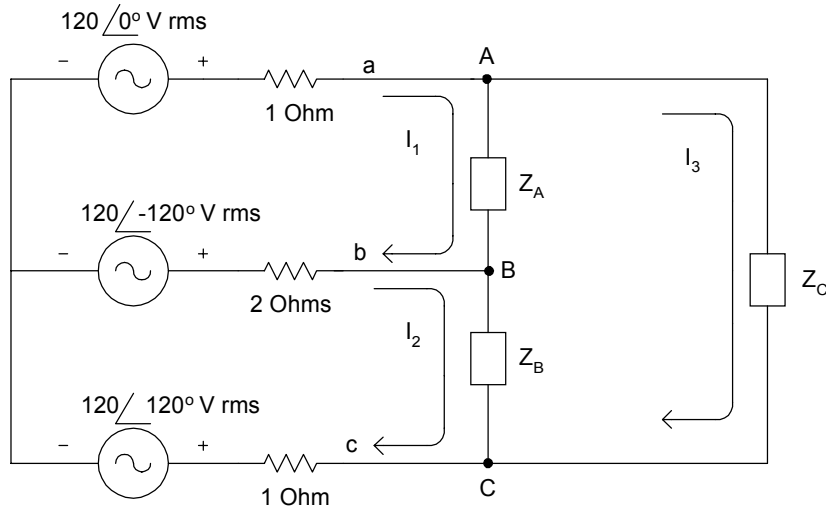

Figure P6.3 Unbalanced Three-phase System

6.4   For the system with network function

$$H(s) = \frac{s^3 + 4s^2 + 16s + 4}{s^4 + 20s^3 + 12s^2 + s + 10}$$

find the poles and zeros of $H(s)$.

6.5   Use MATLAB to determine the roots of the following polynomials. Plot the polynomial over the appropriate interval to verify the roots location.

(a) $f_1(x) = x^2 + 4x + 3$

(b) $f_2(x) = x^3 + 5x^2 + 9x + 5$

(c) $f_3(x) = 2x^5 - 4x^4 - 12x^3 + 27x^2 + 8x - 16$

6.6   If $\dfrac{V_o(s)}{V_i(s)} = \dfrac{20s}{15s^2 + 23s + 16}$,

find $v_0(t)$ given that $v_i(t) = 2.3e^{-2t} \cos(5t + 30^0)$.

6.7   For the circuit of Figure P6.7

(a)   Find the transfer function $\dfrac{V_o(s)}{V_i(s)}$.

(b)   If $v_i(t) = 10e^{-5t} \cos(t + 10^0)$, find $v_0(t)$.
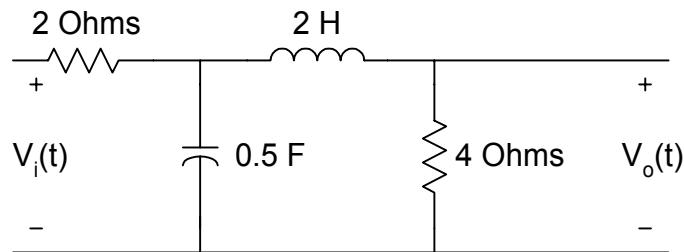

Figure P6.7 RLC Circuit

6.8   For Figure P6.8,

(a) Find the transfer function $H(s) = \dfrac{V_o(s)}{V_i(s)}$.
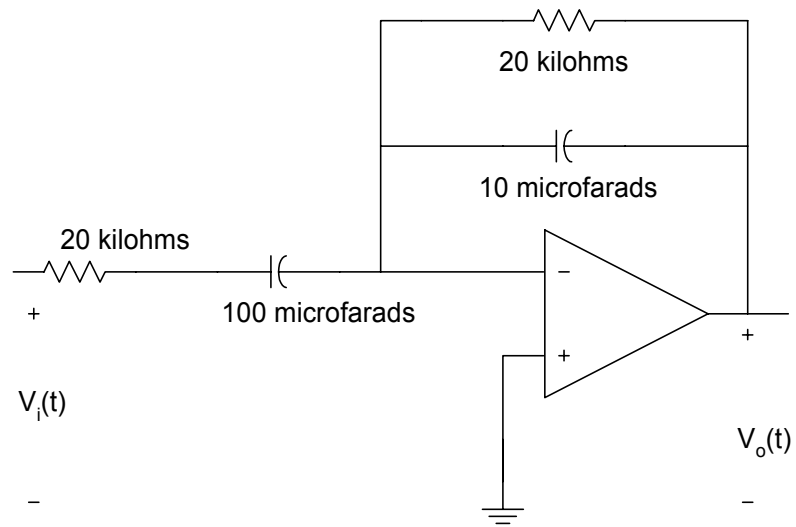
(b) Use MATLAB to plot the magnitude characteristics.

Figure P6.8 Simple Active Filter