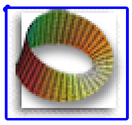

实验12　求矩阵的特征值和特征向量

本文档举例说明计算矩阵特征值和特征向量的方法.

例1　求矩阵 $A := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ 的特征值与特征向量.

调用特征值函数　　$\text{eigenvals}(A) \rightarrow \begin{pmatrix} 0 \\ \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$

$$\text{eigenvecs}(A) \rightarrow \begin{pmatrix} \frac{-1}{2}\cdot\sqrt{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2}\cdot\sqrt{2} & \frac{-1}{2}\cdot\sqrt{2} \\ \frac{1}{2}\cdot\sqrt{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

调用特征向量函数,输出矩阵的每一列向量为对应于各特征值的一个特征向量。

$$\text{eigenvec}(A,0) = \begin{pmatrix} -0.707 \\ 0 \\ 0.707 \end{pmatrix} \quad \text{eigenvec}(A,\sqrt{2}) = \begin{pmatrix} 0.5 \\ 0.707 \\ 0.5 \end{pmatrix} \quad \text{eigenvec}(A,-\sqrt{2}) = \begin{pmatrix} 0.5 \\ -0.707 \\ 0.5 \end{pmatrix}$$

Mathcad 中输出的特征向量是经过标准化成单位长度的。

按照通常的计算过程分步骤计算：设 $I := \text{identity}(3)$

特征多项式　　$|\lambda\cdot I - A| \rightarrow \lambda^3 - 2\cdot\lambda$

调用求多项式根的函数得到特征根　　$\text{polyroots}\left(\begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1.414 \\ 0 \\ 1.414 \end{pmatrix}$

$$\text{rref}(0\cdot I - A) \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \xi1 := \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \frac{\xi1}{|\xi1|}^{T} \rightarrow \left(\frac{-1}{2}\cdot\sqrt{2} \;\; 0 \;\; \frac{1}{2}\cdot\sqrt{2} \right)$$

对特征值 0

$$\text{rref}(\sqrt{2}\cdot I - A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1.414 \\ 0 & 0 & 0 \end{pmatrix} \qquad \xi2 := \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad \frac{\xi2}{|\xi2|}^{T} \rightarrow \left(\frac{1}{2} \;\; \frac{1}{2}\cdot\sqrt{2} \;\; \frac{1}{2} \right)$$

对特征值 $\sqrt{2}$

$$\text{rref}(-\sqrt{2}\cdot I - A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.414 \\ 0 & 0 & 0 \end{pmatrix} \qquad \xi3 := \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \qquad \frac{\xi3}{|\xi3|}^{T} \rightarrow \left(\frac{1}{2} \;\; \frac{-1}{2}\cdot\sqrt{2} \;\; \frac{1}{2} \right)$$

对特征值 $-\sqrt{2}$

$$\sqrt{2}\cdot I - A \rightarrow \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \qquad -\sqrt{2}\cdot I - A \rightarrow \begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix}$$

例2　求矩阵 $A := \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ **的特征值与特征向量**

$\mathrm{eigenvals}(A) \rightarrow \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$　　**调用特征值函数, 2为重根**

$\mathrm{eigenvecs}(A) \rightarrow \begin{pmatrix} \frac{-1}{2}\cdot\sqrt{2} & \frac{-1}{2}\cdot\sqrt{2} & \frac{1}{3}\cdot\sqrt{3} \\ 0 & \frac{1}{2}\cdot\sqrt{2} & \frac{1}{3}\cdot\sqrt{3} \\ \frac{1}{2}\cdot\sqrt{2} & 0 & \frac{1}{3}\cdot\sqrt{3} \end{pmatrix}$　　**调用特征向量函数,输出向量的每一列为对应于各特征值的特征向量。**

$\mathrm{eigenvec}(A,5) = \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix}$　$\mathrm{eigenvec}(A,2) = \begin{pmatrix} -0.615 \\ -0.158 \\ 0.773 \end{pmatrix}$

$\xi1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$　$\frac{\xi1}{|\xi1|}^{T} \rightarrow \begin{pmatrix} \frac{1}{3}\cdot\sqrt{3} & \frac{1}{3}\cdot\sqrt{3} & \frac{1}{3}\cdot\sqrt{3} \end{pmatrix}$　　**对特征值 4 的特征向量**

对特征根1基础解系为：

$\xi2 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$　$\frac{\xi2}{|\xi2|}^{T} \rightarrow \begin{pmatrix} \frac{1}{2}\cdot\sqrt{2} & 0 & \frac{-1}{2}\cdot\sqrt{2} \end{pmatrix}$　$\xi3 := \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$　$\frac{\xi3}{|\xi3|}^{T} \rightarrow \begin{pmatrix} 0 & \frac{1}{2}\cdot\sqrt{2} & \frac{-1}{2}\cdot\sqrt{2} \end{pmatrix}$

$\xi(c1,c2) := c1\xi2 + c2\xi3$　　$\xi(c1,c2) \rightarrow \begin{pmatrix} c1 \\ c2 \\ -c1 - c2 \end{pmatrix}$　　**对特征值1的全部特征向量,** c1 **和** c2 **为任意不全为0的常数**

$c1 := -0.158$　　$c2 := -0.615$　　$-c1 - c2 = 0.773$

$x := \xi(-1,-4)$　$\frac{x}{|x|}^{T} \rightarrow \begin{pmatrix} \frac{-1}{42}\cdot\sqrt{42} & \frac{-2}{21}\cdot\sqrt{42} & \frac{5}{42}\cdot\sqrt{42} \end{pmatrix} = (-0.154 \;\; -0.617 \;\; 0.772)$　　$|x| \rightarrow \sqrt{42}$

$\xi(-1,-4) \rightarrow \begin{pmatrix} -1 \\ -4 \\ 5 \end{pmatrix}$

$Q := \begin{pmatrix} \frac{-1}{2}\cdot\sqrt{2} & \frac{-1}{2}\cdot\sqrt{2} & \frac{1}{3}\cdot\sqrt{3} \\ 0 & \frac{1}{2}\cdot\sqrt{2} & \frac{1}{3}\cdot\sqrt{3} \\ \frac{1}{2}\cdot\sqrt{2} & 0 & \frac{1}{3}\cdot\sqrt{3} \end{pmatrix}$　$Q^{-1}\cdot A\cdot Q \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$　$W := \mathrm{eigenvecs}(A)$　$W^{-1}\cdot A\cdot W \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$