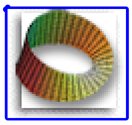


实验5　微积分运算(五)　求重积分运算

本工作页进行多元函数的重积分的实验.

1. **定义多元函数, 事前弄清积分区域.**
2. **将积分表为累次积分形式, 可以使用符号运算(如果有符号解) , 或者求出浮点解. 输入定积分号的热键Shift+7.**
3. **必要时, 可以借助Mathcad的图形功能, 生成区域图形, 直观地确定积分区域.**

例1 求函数 f(x,y) 在{(x,y)| x>0, y>0, x+y<a} 上的积分.

$f(x,y) := 1 - x - y$

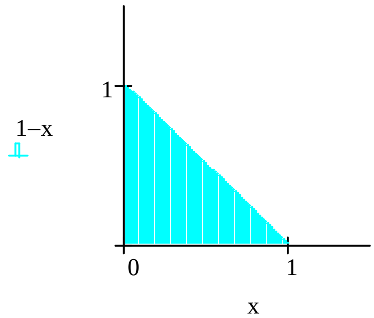
$$\int_0^a \int_0^{a-x} f(x,y)\,dy\,dx \rightarrow \frac{1}{2}\cdot a^2 - \frac{1}{3}\cdot a^3$$

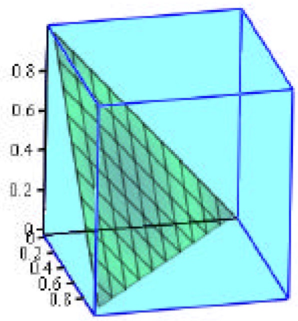
$$\int_0^a \int_0^{a-y} f(x,y)\,dx\,dy \rightarrow \frac{1}{2}\cdot a^2 - \frac{1}{3}\cdot a^3$$

$$\int_0^a \int_0^{a-x} 1 - x - y\,dy\,dx \rightarrow \frac{1}{2}\cdot a^2 - \frac{1}{3}\cdot a^3$$

$$\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy \rightarrow \frac{1}{6}$$

$a := 3$　　$$\int_0^a \int_0^{a-x} 1 - x - y\,dy\,dx \rightarrow \frac{-9}{2}$$



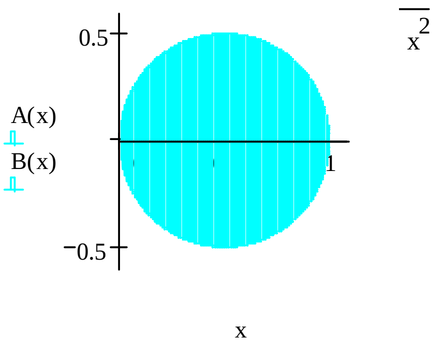


例2 求函数 f(x,y) 在{(x,y)| x^2+y^2<x} 上的积分.

$A(x) := \sqrt{x - x^2}$　　$B(x) := -A(x)$

$g(x,y) := \sqrt{x}$

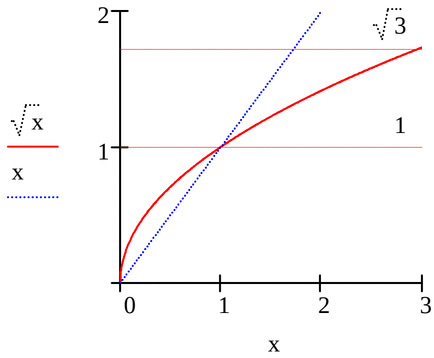
$$I := \int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \sqrt{x}\,dy\,dx \qquad I \rightarrow \frac{8}{15}$$

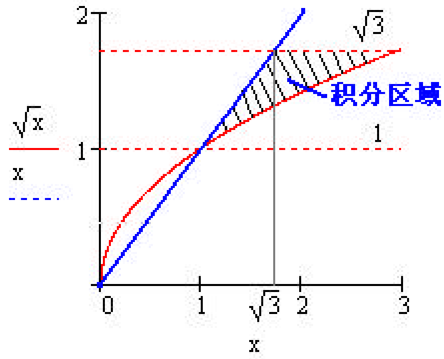


例3 求函数 f(x,y) 在{ (x,y)| y < x < y^2, 1 < y < sqrt(3) } 上的积分.

$$f(x,y) := \frac{y}{x^2 + y^2}$$

在Mathcad中, 当在工作页的下方定义了与上方tong名的函数时, 将自动地复盖前面的函数.



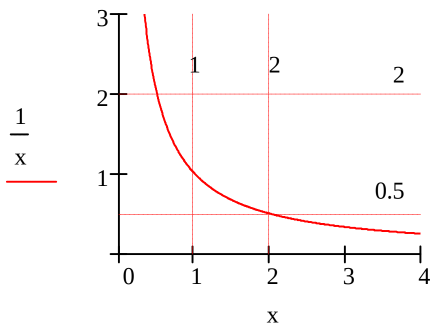


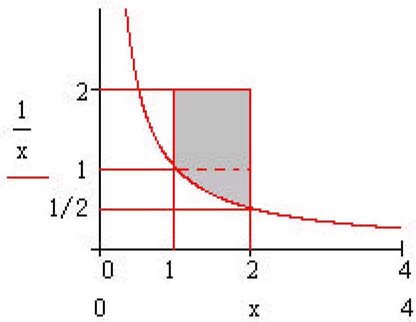
$$I := \int_1^{\sqrt{3}} \int_y^{y^2} f(x,y)\,dx\,dy \qquad I \rightarrow \frac{1}{12}\cdot\pi\cdot\sqrt{3} - \frac{1}{2}\cdot\ln(2) \qquad I = 0.10688$$

$$I1 := \int_1^{\sqrt{3}} \int_{\sqrt{x}}^{x} f(x,y)\,dy\,dx + \int_{\sqrt{3}}^{3} \int_{\sqrt{x}}^{\sqrt{3}} f(x,y)\,dy\,dx \qquad I1 \text{ simplify } \rightarrow \frac{-1}{2}\cdot\ln(2) + \frac{1}{12}\cdot\pi\cdot\sqrt{3}$$

例4 求函数 f(x,y) 在{ (x,y)| 1/x < y < 2, 1 < x < 2 } 上的积分.

$$f(x,y) := y\cdot e^{x\cdot y}$$





$$I := \int_1^2 \int_{\frac{1}{x}}^{2} f(x,y)\,dy\,dx \qquad I \rightarrow \frac{1}{2}\cdot\exp(4) - \exp(2)$$

$$I1 := \int_{\frac{1}{2}}^{1} \int_{\frac{1}{y}}^{2} f(x,y)\,dx\,dy + \int_1^2 \int_1^2 f(x,y)\,dx\,dy \qquad I1 \rightarrow \frac{1}{2}\cdot\exp(4) - \exp(2)$$

例5 求函数u= f(x,y,z) 在{ (x,y,z)| 0 < x < 2, 0 < y <2-x , 0 < z < 2-x-y } 上的积分.

$$f(x,y,z) := \left(x + y^2\right)\cdot z^2 \qquad \int_0^2 \int_0^{2-x} \int_0^{2-x-y} f(x,y,z)\,dz\,dy\,dx \rightarrow \frac{88}{315}$$

例5 求函数u= f(x,y) 的广义积分.

$$\int_0^1 \int_0^x \frac{x+y}{\left(x^2+y^2\right)} \, dy \, dx \rightarrow \frac{1}{2} \cdot \ln(2) + \frac{1}{4} \cdot \pi$$

$$\int_0^1 \int_0^x \frac{x \cdot y}{\left(x^2+y^2\right)} \, dy \, dx \rightarrow \frac{1}{4} \cdot \ln(2)$$

$$\int_0^c \int_{-\sqrt{c^2-x^2}}^{\sqrt{c^2-x^2}} \frac{x \cdot y^2}{\sqrt{c^2-x^2}} \, dy \, dx \rightarrow \frac{1}{6} \cdot c^4$$

$$\int_0^1 \int_x^1 \frac{x}{x^2+y^2} \, dy \, dx \rightarrow \frac{1}{2} \cdot \ln(2)$$

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}} \, dy \, dx \rightarrow 2 \cdot \pi$$

$$\int_1^\infty \int_1^\infty \frac{1}{x^2 \cdot y^2} \, dy \, dx \rightarrow 1$$

$$\int_0^\infty \int_x^\infty x \cdot y \cdot e^{-\left(x^2+y^2\right)} \, dy \, dx \rightarrow \frac{1}{8}$$

$$\int_0^\infty \int_0^\infty \frac{1}{(x+y+1)^3} \, dy \, dx \rightarrow \frac{1}{2}$$

以 $\int_0^1 \int_x^1 \frac{x}{x^2+y^2} \, dy \, dx$ **为例, Mathcad中, 计算重积分的过程可分解为:**

$$\int_x^1 \frac{x}{x^2+y^2} \, dy \rightarrow \operatorname{atan}\left(\frac{1}{x}\right) - \frac{1}{4} \cdot \pi$$

$$\int_0^1 \operatorname{atan}\left(\frac{1}{x}\right) - \frac{1}{4} \cdot \pi \, dx \rightarrow \frac{1}{2} \cdot \ln(2)$$

在计算广义积分 $\int_0^1 \int_0^1 \frac{x \cdot y}{\left(x^2+y^2\right)^{\frac{3}{2}}} \, dy \, dx \rightarrow$ **不能得到符号解. 系统提示**

Can't divide by zero 表示在运算中出现了0作除数的情况. 分解成两步来求解可以得到: 内层积分为

$$\int_0^1 \frac{x \cdot y}{\left(x^2+y^2\right)^{\frac{3}{2}}} \, dy \text{ simplify } \rightarrow \frac{1}{x^2} \cdot \frac{\operatorname{csgn}(x)}{\left[\frac{\left(x^2+1\right)}{x^2}\right]^{\frac{1}{2}} \cdot \left[1 + \left[\frac{\left(x^2+1\right)}{x^2}\right]^{\frac{1}{2}}\right]}$$

$$\int_0^1 \frac{1}{x^2} \cdot \frac{\operatorname{csgn}(x)}{\left[\frac{\left(x^2+1\right)}{x^2}\right]^{\frac{1}{2}} \cdot \left[1 + \left[\frac{\left(x^2+1\right)}{x^2}\right]^{\frac{1}{2}}\right]} \, dx \rightarrow$$

将被积表达式改写成: $\dfrac{\operatorname{csgn}(x)}{\sqrt{x^2+1}\cdot\left(x+\sqrt{x^2+1}\right)}$ **,对这个函数求积分:**

$$\int_0^1 \frac{\operatorname{csgn}(x)}{\sqrt{x^2+1}\cdot\left[x+\left(\sqrt{x^2+1}\right)\right]}\,dx \to 2-\sqrt{2} \qquad \int_0^1\int_0^1 \frac{x\cdot y}{\left(x^2+y^2\right)^{\frac{3}{2}}}\,dy\,dx = 2-\sqrt{2}$$