实验6　　微积分运算(六)　　重积分运算, 积分变换

本工作页继续进行多元函数的重积分的实验.

1. **用极坐标变换求积分. Polar coordinates**
2. **用球坐标变换求积分. Sphere coordinates**
3. **用柱坐标变换求积分. Cylindrical coordinates**

1 变换的Jackob行列式

(1) 极坐标变换

$$Rp(r,\theta) := \begin{pmatrix} r\cdot\cos(\theta) \\ r\cdot\sin(\theta) \end{pmatrix} \qquad Jp(r,\theta) := \left|\begin{pmatrix} \frac{d}{dr}Rp(r,\theta)_0 & \frac{d}{dr}Rp(r,\theta)_1 \\ \frac{d}{d\theta}Rp(r,\theta)_0 & \frac{d}{d\theta}Rp(r,\theta)_1 \end{pmatrix}\right| \qquad Jpolar(r,\theta)\ \text{simplify}\ \rightarrow Jpolar(r,\theta)$$

(2) 球坐标变换

$$Rs(r,\phi,\varphi) := \begin{pmatrix} r\cdot\sin(\phi)\cdot\cos(\varphi) \\ r\cdot\sin(\phi)\cdot\sin(\varphi) \\ r\cdot\cos(\phi) \end{pmatrix} \qquad Js(r,\phi,\varphi) := \left|\begin{pmatrix} \frac{d}{dr}Rs(r,\phi,\varphi)_0 & \frac{d}{dr}Rs(r,\phi,\varphi)_1 & \frac{d}{dr}Rs(r,\phi,\varphi)_2 \\ \frac{d}{d\phi}Rs(r,\phi,\varphi)_0 & \frac{d}{d\phi}Rs(r,\phi,\varphi)_1 & \frac{d}{d\phi}Rs(r,\phi,\varphi)_2 \\ \frac{d}{d\varphi}Rs(r,\phi,\varphi)_0 & \frac{d}{d\varphi}Rs(r,\phi,\varphi)_1 & \frac{d}{d\varphi}Rs(r,\phi,\varphi)_2 \end{pmatrix}\right|$$

$$Js(r,\phi,\varphi)\ \text{simplify}\ \rightarrow \sin(\phi)\cdot r^2$$

(3) 柱坐标变换

$$Rc(r,\phi,z) := \begin{pmatrix} r\cdot\cos(\phi) \\ r\cdot\sin(\phi) \\ z \end{pmatrix} \qquad Jc(r,\phi,z) := \left|\begin{pmatrix} \frac{d}{dr}Rc(r,\phi,z)_0 & \frac{d}{dr}Rc(r,\phi,z)_1 & \frac{d}{dr}Rc(r,\phi,z)_2 \\ \frac{d}{d\phi}Rc(r,\phi,z)_0 & \frac{d}{d\phi}Rc(r,\phi,z)_1 & \frac{d}{d\phi}Rc(r,\phi,z)_2 \\ \frac{d}{dz}Rs(r,\phi,z)_0 & \frac{d}{dz}Rc(r,\phi,z)_1 & \frac{d}{dz}Rc(r,\phi,z)_2 \end{pmatrix}\right|$$

$$Jc(r,\phi,z)\ \text{simplify}\ \rightarrow r$$

2 利用以上各种变换计算重积分.

例1 计算二重积分, 积分域D={(x,y)|x^2+y^2<1}

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dy\,dx \rightarrow \int_{-1}^{1} \ln\left[\left[-(-1+x)\cdot(x+1)\right]^{\frac{1}{2}}+\sqrt{2}\right]-\ln\left[-\left[-(-1+x)\cdot(x+1)\right]^{\frac{1}{2}}+\sqrt{2}\right]dx = 2.6$$

在直角坐标下求不出符号解. 应用极坐标变换:

$$\int_{0}^{2\cdot\pi}\int_{0}^{1} \frac{r}{\sqrt{1+r^2}}\,dr\,d\theta \rightarrow 2\cdot\sqrt{2}\cdot\pi - 2\cdot\pi = 2.60258$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dy\,dx = 1.793$$

无符号解

$$\int_{0}^{2\cdot\pi}\int_{0}^{1}\sqrt{\frac{1-r^2}{1+r^2}}\cdot r\,dr\,d\theta \to \frac{1}{2}\cdot\pi^2-\pi = 1.793$$

$$r^2 = t \quad 2r\cdot dr = dt \quad \sqrt{\frac{1-r^2}{1+r^2}}\ \text{substitute}, r = \sqrt{t} \ \to \left[\frac{(1-t)}{(1+t)}\right]^{\frac{1}{2}} \quad \int_{0}^{2\pi}\int_{0}^{1}\left[\frac{(1-t)}{(1+t)}\right]^{\frac{1}{2}}\cdot\frac{1}{2}\,dt\,d\theta \to \frac{1}{2}\cdot\pi^2-\pi$$

在圆环域1<x^2+y^2<4上计算 $\ln\left(x^2+y^2\right)$ 的积分.应用极坐标.

$$\int_{0}^{2\pi}\int_{1}^{2}\ln\left(r^2\right)\cdot r\,dr\,d\theta \to 8\cdot\pi\cdot\ln(2)-3\cdot\pi$$

例2 计算三重积分

(1) 计算椭球体 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ 的体积.

$$V(a,b,c) := 8\cdot a\cdot b\cdot c\cdot\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\int_{0}^{1}r^2\cdot\sin(\phi)\,dr\,d\phi\,d\varphi \qquad V(a,b,c) \to \frac{4}{3}\cdot a\cdot b\cdot c\cdot\pi$$

直接在直角坐标下求出的符号解为:

$$8\cdot\int_{0}^{a}\int_{0}^{b\cdot\sqrt{1-\frac{x^2}{a^2}}}\int_{0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}1\,dz\,dy\,dx \to \frac{4}{3}\cdot c\cdot a\cdot\frac{\left(2\cdot\ln(b)+\ln\left(\frac{-1}{b^2}\right)\right)}{\left(\frac{-1}{b^2}\right)^{\frac{1}{2}}}$$

$$x(a,\rho,\theta) := a\cdot\rho\cdot\cos(\theta) \qquad y(b,\rho,\theta) := b\cdot\rho\cdot\sin(\theta) \qquad J(a,b,\rho,\theta) := a\cdot b\cdot\rho$$

$$V2(a,b,c) := 8\cdot\int_{0}^{\frac{\pi}{2}}\int_{0}^{1}c\cdot\sqrt{1-\rho^2}\cdot a\cdot b\cdot\rho\,d\rho\,d\theta \qquad V2(a,b,c) \to \frac{4}{3}\cdot\pi\cdot c\cdot a\cdot b$$

(2) 计算函数u=xyz在D={(x,y)| x^2 +y^2+z^2<a^2 ,a>0}上的积分

$$\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}}x\cdot y\cdot z\,dz\,dy\,dx \to \frac{1}{48}\cdot a^6 \qquad x\cdot y\cdot z = r^3\cdot\sin(\phi)\cdot\cos(\phi)\cdot\sin(\theta)^2\cdot\cos(\theta) \qquad J = r^2\cdot\sin(\theta)$$

$$\int_{0}^{a}\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}r^5\cdot\sin(\phi)\cdot\cos(\phi)\cdot\sin(\theta)^3\cdot\cos(\theta)\,d\theta\,d\phi\,dr \to \frac{1}{48}\cdot a^6$$

(3) 计算函数 $u = \dfrac{x^2+y^2+z^2}{\sqrt{1-x^2-y^2-z^2}}$ 在D={(x,y)| x^2 +y^2+z^2<1}上的积分

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\frac{r^2}{\sqrt{1-r^2}}\cdot r^2\cdot\sin(\theta)\,dr\,d\theta\,d\phi \rightarrow \frac{3}{4}\cdot\pi^2$$

计算函数 $u = z^2 - x^2 - y^2$ 在D={(x,y)| x^2 +y^2<a^2, |z|<a}上的积分, 应用柱坐标变换.

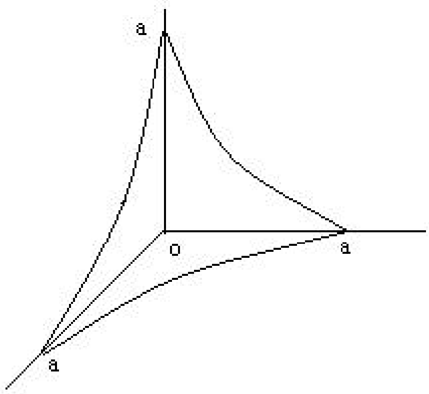
$$I(a) := \int_0^{2\pi}\int_0^{a}\int_{-a}^{a}\left(z^2-r^2\right)\cdot r\,dz\,dr\,d\phi \qquad I(a) \rightarrow \frac{-1}{3}\cdot\pi\cdot a^5$$

(4) 计算x^(2/3)+y^(2/3)+z^(2/3)<a^(2/3)所界定的闭域的体积.
右图为该立体在第一卦线中的一部分.

$x = a\cdot u^3 \qquad y = a\cdot v^3 \qquad z = a\cdot w^3$

$$V(a) := 8\cdot 27\cdot a^3\int_0^1\int_0^{\sqrt{1-u^2}}\int_0^{\sqrt{1-u^2-v^2}} u^2\cdot v^2\cdot w^2\,dw\,dv\,du$$

$$V(a) \rightarrow \frac{4}{35}\cdot a^3\cdot\pi$$



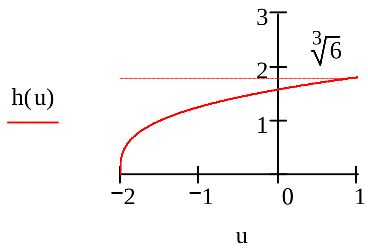
再引入球坐标

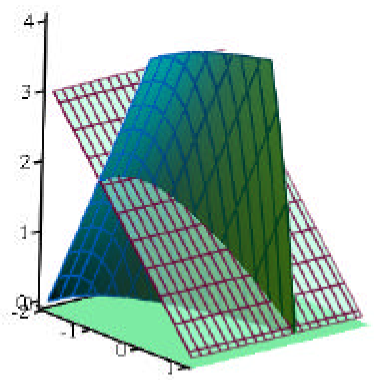
$$V(a) := 27\cdot a^3\cdot\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^8\cdot\sin(\phi)^5\cdot\cos(\phi)^2\cdot\sin(\theta)^2\cdot\cos(\theta)^2\,dr\,d\phi\,d\theta \qquad V(a) \rightarrow \frac{4}{35}\cdot a^3\cdot\pi$$

(5) 计算由曲面y^3=2x+4和平面x+z=1,z=0所围成的立体体积.

$h(x) := \sqrt[3]{(2\cdot x+4)}$

$f(x,y) := 1 - x \qquad g(x,y) := 2\cdot x - y^3 + 4$





f, g

曲面f ,g 以及xOy平面所围成的立体的体积如右图所示.

$$V := \int_0^{\sqrt[3]{6}} \int_{\frac{y^3-4}{2}}^{1} \int_0^{1-x} 1 \, dz \, dx \, dy \qquad V \to \frac{81}{28} \cdot \sqrt[3]{6} \qquad V = 5.257$$

$$V1 := \int_{-2}^{1} \int_0^{\sqrt[3]{2x+4}} \int_0^{1-x} 1 \, dz \, dy \, dx \qquad V1 \to \frac{81}{28} \cdot \sqrt[3]{6} \qquad V2 := \int_0^{3} \int_{-2}^{1-z} \int_0^{\sqrt[3]{2x+4}} 1 \, dy \, dx \, dz \qquad V2 \to \frac{81}{28} \cdot \sqrt[3]{6}$$