

Chapter 1 Electromagnetic Field Theory

1-1 Electric Fields and Electric Dipoles

Gauss's law of \vec{E} : $\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon} = \frac{1}{\epsilon} \iiint_V \rho dv'$

$\xrightarrow{\text{divergence theorem}} \iiint_V \nabla \cdot \vec{E} dv' = \frac{1}{\epsilon} \iiint_V \rho dv' \Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon}$ and $\epsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ (F/m) in the free space.

For q at \vec{R}' , field point at $\vec{R} \Rightarrow \vec{E} = \frac{q \hat{a}_{RR'}}{4\pi\epsilon |\vec{R} - \vec{R}'|^2} = \frac{q(\vec{R} - \vec{R}')}{4\pi\epsilon |\vec{R} - \vec{R}'|^3}$ and $\hat{a}_{RR'} = \frac{\vec{R} - \vec{R}'}{|\vec{R} - \vec{R}'|}$.

\vec{E} due to a system of discrete charges: $\vec{E} = \frac{1}{4\pi\epsilon} \sum_{k=1}^n \frac{q_k (\vec{R} - \vec{R}'_k)}{|\vec{R} - \vec{R}'_k|^3}$

Volume source $\rho \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon} \iiint_{V'} \hat{a}_R \frac{\rho}{R^2} dv' = \frac{1}{4\pi\epsilon} \iiint_{V'} \frac{\rho \vec{R}}{|\vec{R}|^3} dv'$

Surface source $\rho_s \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon} \iint_{S'} \hat{a}_R \frac{\rho_s}{R^2} dS'$ Line source $\rho_l \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon} \int_{l'} \hat{a}_R \frac{\rho_l}{R^2} dl'$

Eg. Show that Coulomb's law $\vec{F} = \hat{a}_R \frac{q_1 q_2}{4\pi\epsilon R^2} = \frac{q_1 q_2 \vec{R}}{4\pi\epsilon R^3}$, where $\hat{a}_R = \frac{\vec{R}}{R}$, $R = |\vec{R}|$.

(Proof) $\because \vec{F} = q_2 \vec{E}$ and $\oiint \vec{E} \cdot d\vec{S} = \frac{q_1}{\epsilon} = 4\pi r^2 E$, $\vec{E} = \hat{a}_R \frac{q_1}{4\pi\epsilon R^2} = \frac{q_1 \vec{R}}{4\pi\epsilon R^3}$

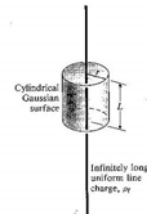
$$\therefore \vec{F} = \hat{a}_R \frac{q_1 q_2}{4\pi\epsilon R^2}$$

Eg. Determine the electric field intensity of an infinitely long line charge of a uniform density ρ_l in air.

(Sol.)

$$\oiint_S \vec{E} \cdot d\vec{S} = \int_0^L \int_0^{2\pi} E r d\phi dz = 2\pi r L E$$

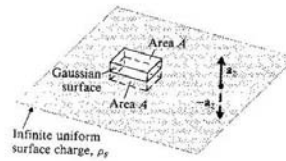
$$2\pi r L E = \frac{\rho_l L}{\epsilon_0}, \quad \vec{E} = \hat{a}_r \frac{\rho_l}{2\pi\epsilon_0 r}$$



Eg. Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s .

(Sol.) $\oint_S \vec{E} \cdot d\vec{S} = 2ES = 2EA, \quad 2EA = \frac{\rho_s A}{\epsilon_0}$

$$\Rightarrow \vec{E} = \begin{cases} = \hat{z} \frac{\rho_s}{2\epsilon_0}, & z > 0 \\ = -\hat{z} \frac{\rho_s}{2\epsilon_0}, & z < 0 \end{cases}$$



Eg. A line charge of uniform density ρ_l in free space forms a semicircle of radius b . Determine the magnitude and direction of the electric field intensity at the center of the semicircle. [高考]

(Sol.) $d\vec{E}_y = -\frac{\rho_l(b d\phi)}{4\pi\epsilon_0 b^2} \cdot \sin\phi, \quad \vec{E} = \hat{y} E_y = -\hat{y} \frac{\rho_l}{4\pi\epsilon_0 b} \int_0^\pi \sin\phi d\phi = -\hat{y} \frac{\rho_l}{2\pi\epsilon_0 b}$

Eg. Determine the electric field caused by spherical cloud of electrons with a volume charge density $\rho = -\rho_0$ for $0 \leq R \leq b$ (both ρ_0 and b are positive) and $\rho = 0$ for $R > b$. [交大電子物理所]

(Sol.)

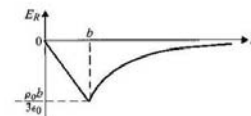
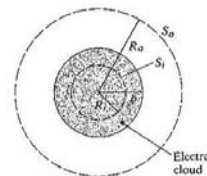
(a) $R \geq b$

$$Q = -\rho_0 \frac{4\pi}{3} b^3, \quad \vec{E} = \hat{a}_R \frac{Q}{4\pi\epsilon_0 R^2} = -\hat{a}_R \frac{\rho_0 b^3}{3\epsilon_0 R^2}$$

(b) $0 \leq R \leq b$

$$\vec{E} = \hat{a}_R E, \quad d\vec{S} = \hat{a}_R ds, \quad \oint_{S_i} \vec{E} \cdot d\vec{S} = E \int_{S_i} dS = E 4\pi R^2$$

$$Q = \iiint_V \rho dv = -\rho_0 \iiint_V dv = -\rho_0 \frac{4\pi}{3} R^3, \quad \vec{E} = -\hat{a}_R \frac{\rho_0 R}{3\epsilon_0}$$

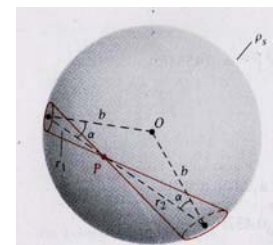


Eg. A total charge Q is put on a thin spherical shell of radius b . Determine the electrical field intensity at an arbitrary point inside the shell. [台大電研]

(Sol.)

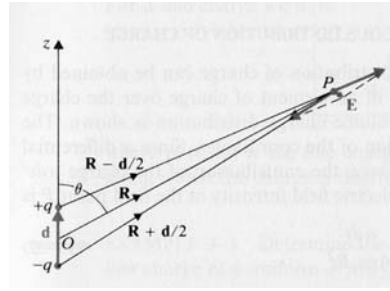
$$\rho_s = \frac{Q}{4\pi b^2}, \quad dE = \frac{\rho_s}{4\pi\epsilon_0} \left(\frac{dS_1}{r_1^2} - \frac{dS_2}{r_2^2} \right)$$

$$d\Omega = \frac{dS_1}{r_1^2} \cos\alpha = \frac{dS_2}{r_2^2} \cos\alpha, \quad dE = \frac{\rho_s}{4\pi\epsilon_0} \left(\frac{d\Omega}{\cos\alpha} - \frac{d\Omega}{\cos\alpha} \right) = 0$$



Electric dipole: A pair of equal but opposite charges with separation.

$$\begin{aligned} \left| \vec{R} - \frac{\vec{d}}{2} \right|^{-3} &= \left[\left(\vec{R} - \frac{\vec{d}}{2} \right) \cdot \left(\vec{R} - \frac{\vec{d}}{2} \right) \right]^{-3/2} = \left[R^2 - R \cdot d + \frac{d^2}{4} \right]^{-3/2} \\ &\cong R^{-3} \left[1 - \frac{\vec{R} \cdot \vec{d}}{R^2} \right]^{-3/2} \cong R^{-3} \left[1 + \frac{3 \vec{R} \cdot \vec{d}}{2 R^2} \right] \end{aligned}$$



$$\left| \vec{R} + \frac{\vec{d}}{2} \right|^{-3} \cong R^{-3} \left[1 - \frac{3 \vec{R} \cdot \vec{d}}{2 R^2} \right]$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon} \left\{ \frac{\vec{R} - \frac{\vec{d}}{2}}{\left| \vec{R} - \frac{\vec{d}}{2} \right|^3} - \frac{\vec{R} + \frac{\vec{d}}{2}}{\left| \vec{R} + \frac{\vec{d}}{2} \right|^3} \right\} \cong \frac{q}{4\pi\epsilon R^3} \left[3 \frac{\vec{R} \cdot \vec{d}}{R^2} \vec{R} - \vec{d} \right] = \frac{1}{4\pi\epsilon R^3} \left[3 \frac{\vec{R} \cdot \vec{P}}{R^2} \vec{R} - \vec{P} \right]$$

$$\left(\begin{array}{l} \vec{p} = \hat{z} p = p \left(\hat{a}_r \cos \theta - \hat{a}_\theta \sin \theta \right) \\ \vec{R} \cdot \vec{p} = p \cos \theta \end{array} \right)$$

$$\Rightarrow \vec{E} = \frac{p}{4\pi\epsilon R^3} \left(\hat{a}_r 2 \cos \theta + \hat{a}_\theta \sin \theta \right)$$

Eg. At what value of θ does the electric field intensity of a z -directed dipole have no z -component.

$$\text{(Sol.) } \vec{E} = \frac{p}{4\pi\epsilon_0 R^3} (\hat{a}_r 2 \cos \theta + \hat{a}_\theta \sin \theta), \quad \hat{z} = \hat{a}_r \cos \theta - \hat{a}_\theta \sin \theta$$

No z -component $\Rightarrow 2 \cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \theta = 0 \Rightarrow \tan^2 \theta = 2 \Rightarrow \theta = 54.7^\circ$ or 125.3°

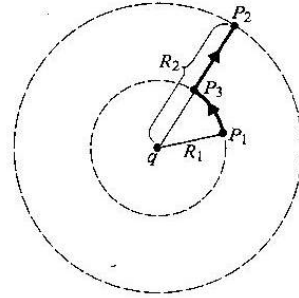
1-2 Static Electric Potentials

$$\vec{E} = -\nabla V \Rightarrow V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} \quad \text{and} \quad \nabla^2 V = -\rho/\epsilon$$

Electric potential due to a point charge:

$$V = -\int_{\infty}^R \hat{a}_R \frac{q}{4\pi\epsilon R^2} \cdot \hat{a}_R dR = \frac{q}{4\pi\epsilon R}$$

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon} \left(\frac{1}{R_2} - \frac{1}{R_1} \right)$$



Electric potential due to discrete charges:

$$V = \frac{1}{4\pi\epsilon} \sum_{k=1}^n \frac{q_k}{|R - R'_k|}$$

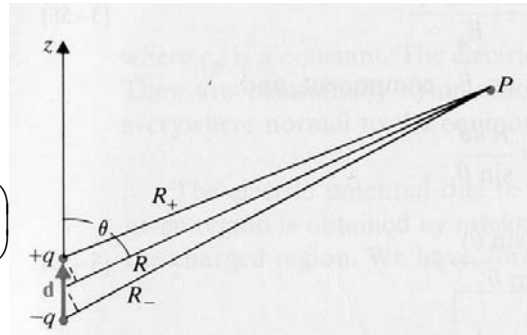
Electric potential due to an electric dipole:

$$V = \frac{q}{4\pi\epsilon} \left(\frac{1}{R_+} - \frac{1}{R_-} \right)$$

If $d \ll R$, we have

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2} \cos \theta \right)^{-1} \cong R^{-1} \left(1 + \frac{d}{2R} \cos \theta \right)$$

and $\frac{1}{R_-} \cong \left(R + \frac{d}{2} \cos \theta \right)^{-1} \cong R^{-1} \left(1 - \frac{d}{2R} \cos \theta \right)$



$$V = \frac{qd \cos \theta}{4\pi\epsilon R^2}, \quad \vec{p} = q\vec{d} \Rightarrow V = \frac{\vec{p} \cdot \hat{a}_R}{4\pi\epsilon R^2} \quad (V)$$

$$\Rightarrow \vec{E} = -\nabla V = -\hat{a}_R \frac{\partial V}{\partial R} - \hat{a}_\theta \frac{\partial V}{\partial \theta} = \frac{p}{4\pi\epsilon R^3} (\hat{a}_R 2 \cos \theta + \hat{a}_\theta \sin \theta)$$

Scalar electric potential due to various charge distributions:

Volume source $\rho \Rightarrow V = \frac{1}{4\pi\epsilon} \iiint_{v'} \frac{\rho}{R} dv'$

Surface source $\rho_s \Rightarrow V = \frac{1}{4\pi\epsilon} \iint_{s'} \frac{\rho_s}{R} dS'$

Line source $\rho_l \Rightarrow V = \frac{1}{4\pi\epsilon} \int \frac{\rho_l}{R} dl'$

Note: 1. V is a scalar, but \vec{E} is a vector.

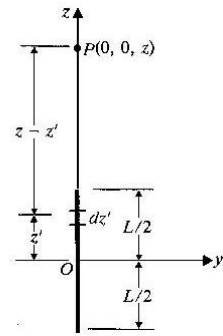
2. $\vec{E} = -\nabla V$ is valid only in the static EM field.

Eg. Obtain a formula for electrical field intensity along the axis of a uniform line charge of length L . The uniform line-charge density is ρ_l . [高考]

(Sol.) $R = z - z', \quad z > \frac{L}{2}$

$$V = \frac{\rho_l}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dz'}{z - z'} = \frac{\rho_l}{4\pi\epsilon_0} \ln \left[\frac{z + (L/2)}{z - (L/2)} \right], \quad z > \frac{L}{2}$$

$$\vec{E} = -\hat{z} \frac{dV}{dz} = \hat{z} \frac{\rho_l L}{4\pi\epsilon_0 [z^2 - (L/2)^2]}, \quad z > \frac{L}{2}$$



Eg. A finite line charge of length L carrying uniform line charge density ρ_l is coincident with the x -axis. Determine V and \vec{E} in the plane bisecting the line charge.

(Sol.) $V = \int_{-L/2}^{L/2} \frac{\rho_l dx}{4\pi\epsilon_0 \sqrt{x^2 + y^2}} = \frac{\rho_l}{2\pi\epsilon_0} \left\{ \ln \left[\sqrt{\left(\frac{L}{2}\right)^2 + y^2} + \frac{L}{2} \right] - \ln y \right\}$

and $\vec{E} = -\nabla V = \hat{y} \frac{\rho_l}{2\pi\epsilon_0 y} \left[\frac{L/2}{\sqrt{(L/2)^2 + y^2}} \right]$

Eg. A charge is distributed uniformly over an $L \times L$ square plate. Determine V and \vec{E} at a point on the axis perpendicular to the plate and through its center.

(Sol.) $\rho_s = \frac{Q}{L^2}, \quad y^2 \rightarrow y^2 + z^2, \quad V = \frac{\rho_s}{2\pi\epsilon_0} \int_{-L/2}^{L/2} \left\{ \ln \left[\sqrt{\left(\frac{L}{2}\right)^2 + y^2 + z^2} + \frac{L}{2} \right] - \ln \sqrt{y^2 + z^2} \right\} dy$

$$= \frac{Q}{\pi\epsilon_0 L^2} \left\{ \frac{L}{2} \ln \left[\frac{\sqrt{2\left(\frac{L}{2}\right)^2 + z^2} + \frac{L}{2}}{\sqrt{2\left(\frac{L}{2}\right)^2 + z^2} - \frac{L}{2}} \right] - z \cdot \tan^{-1} \left[\frac{\left(\frac{L}{2}\right)^2}{z \sqrt{2\left(\frac{L}{2}\right)^2 + z^2}} \right] \right\}$$

$$\vec{E} = -\nabla V = \hat{z} \frac{Q}{\pi\epsilon_0 L^2} \tan^{-1} \left[\frac{\left(\frac{L}{2}\right)^2}{z \sqrt{2\left(\frac{L}{2}\right)^2 + z^2}} \right]$$

Eg. A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o . Determine \vec{E} and V as functions of the radial distance R . [高考]

$$\text{(Sol.) } R > R_o, \quad \oiint_s \vec{E} \cdot d\vec{S} = E4\pi R^2 = \frac{Q}{\epsilon_0}, \quad E = \frac{Q}{4\pi\epsilon_0 R^2}, \quad V = -\int_{\infty}^R E dR = \frac{Q}{4\pi\epsilon_0 R}$$

$$R_i < R < R_o, \quad E = 0, \quad V = V \Big|_{R=R_o} = \frac{Q}{4\pi\epsilon_0 R_o}$$

$$R < R_i, \quad E = \frac{Q}{4\pi\epsilon_0 R^2}, \quad V = -\int E dR + C = \frac{Q}{4\pi\epsilon_0 R} + C$$

$$C = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R_o} - \frac{1}{R_i} \right) \Rightarrow V = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R} + \frac{1}{R_o} - \frac{1}{R_i} \right)$$

Eg. A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h . Determine \vec{E} and V on its axis (a) at a point outside the tube, (b) at a point inside the tube.

$$\text{(Sol.) } dV = \int_0^{2\pi} \frac{\rho_s b d\phi' dz'}{4\pi\epsilon_0 \sqrt{b^2 + (z-z')^2}} = \frac{\rho_s b dz'}{2\epsilon_0 \sqrt{b^2 + (z-z')^2}}$$

$$\text{(a) } V_o = \int_0^h \frac{\rho_s b dz'}{2\epsilon_0 \sqrt{b^2 + (z-z')^2}} = \frac{b\rho_s}{2\epsilon_0} \ln \frac{b + \sqrt{b^2 + z^2}}{(z-h) + \sqrt{b^2 + (z-h)^2}}, \quad \rho_s = \frac{Q}{2\pi b h}$$

$$\vec{E}_o = -\hat{z} \frac{dV}{dz} = \hat{z} \frac{b\rho_s}{2\epsilon_0} \left[\frac{1}{\sqrt{b^2 + (z-h)^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right]$$

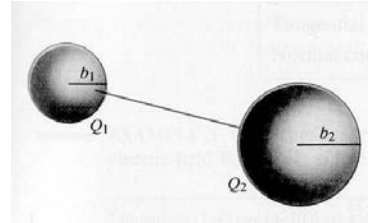
$$\begin{aligned} \text{(b) } V_i &= \int_0^z \frac{\rho_s b dz'}{2\epsilon_0 \sqrt{b^2 + (z-z')^2}} + \int_z^h \frac{\rho_s b dz'}{2\epsilon_0 \sqrt{b^2 + (z'-z)^2}} \\ &= \frac{\rho_s b}{2\epsilon_0} \ln \frac{\rho_s}{b^2} \left(z + \sqrt{b^2 + z^2} \right) \cdot \left[(h-z) + \sqrt{b^2 + (h-z)^2} \right] \end{aligned}$$

Eg. Consider two spherical conductors with radii b_1 and b_2 ($b_2 > b_1$) that are connected wire. The distance between the conductors is very large in comparison to b_2 so that charges on spherical conductors may be considered uniformly distributed. A total charge Q is deposited on the spheres. Find (a) the charges on the two spheres, and (b) the electric field intensities at the sphere surfaces?

(Sol.)

$$(a) \frac{Q_1}{4\pi\epsilon_0 b_1} = \frac{Q_2}{4\pi\epsilon_0 b_2}, \quad \frac{Q_1}{Q_2} = \frac{b_1}{b_2}, \quad Q_1 + Q_2 = Q$$

$$Q_1 = \frac{b_1}{b_1 + b_2} Q \quad \text{and} \quad Q_2 = \frac{b_2}{b_1 + b_2} Q$$

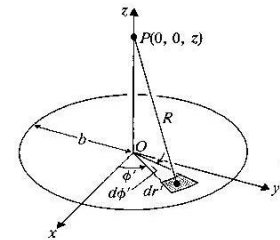


$$(b) E_{1n} = \frac{Q_1}{4\pi\epsilon_0 b_1^2} \quad \text{and} \quad E_{2n} = \frac{Q_2}{4\pi\epsilon_0 b_2^2}, \quad \frac{E_1}{E_2} = \left(\frac{b_2}{b_1}\right)^2 \frac{Q_1}{Q_2} = \frac{b_2}{b_1}$$

Eg. Obtain a formula for the electric field intensity on the axis of a circular disk of radius b that carries uniform surface charge density ρ_s . [高考]

(Sol.) $ds' = r' dr' d\phi'$, $R = \sqrt{z^2 + r'^2}$

$$V = \frac{\rho_s}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^b \frac{r'}{(z^2 + r'^2)^{3/2}} dr' d\phi' = \frac{\rho_s}{2\epsilon_0} \left[(z^2 + b^2)^{1/2} - |z| \right]$$



$$\vec{E} = -\nabla V = -\hat{z} \frac{\partial V}{\partial z} = \begin{cases} \hat{z} \frac{\rho_s}{2\epsilon_0} \left[1 - z(z^2 + b^2)^{-1/2} \right] & z > 0 \\ -\hat{z} \frac{\rho_s}{2\epsilon_0} \left[1 + z(z^2 + b^2)^{-1/2} \right] & z < 0 \end{cases}$$

$$\text{As } z \gg 1 \Rightarrow z(z^2 + b^2)^{-1/2} \cong 1 - \frac{b^2}{2z^2}, \quad \vec{E} = \hat{z} \frac{(\pi b^2 \rho_s)}{4\pi\epsilon_0 z^2} = \begin{cases} \hat{z} \frac{Q}{4\pi\epsilon_0 z^2} & z > 0 \\ -\hat{z} \frac{Q}{4\pi\epsilon_0 z^2} & z < 0 \end{cases}$$

Eg. Make a two-dimensional sketch of the equipotential lines and the electric field lines for an electric dipole.

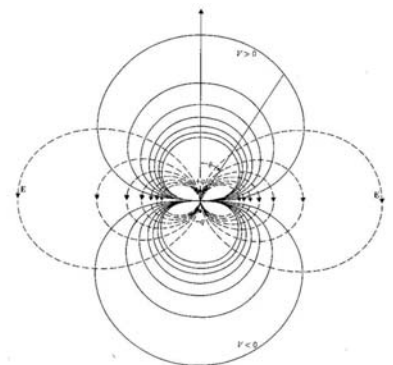
(Sol.)

For an electric dipole, $V = \frac{qd \cos\theta}{4\pi\epsilon_0 R^2} = \text{constant} \Rightarrow R = c_v \sqrt{\cos\theta}$

$d\vec{l} = k\vec{E}$, where k is a constant.

$$\hat{a}_R dR + \hat{a}_\theta R d\theta + \hat{a}_\phi R \sin\theta d\phi = k \left(\hat{a}_R E_R + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi \right)$$

$$\frac{dR}{E_R} = \frac{R d\theta}{E_\theta} = \frac{R \sin\theta d\phi}{E_\phi}, \quad \frac{dR}{2 \cos\theta} = \frac{R d\theta}{\sin\theta}, \quad R = c_E \sin^2 \theta$$



1-3 Magnetic Fields

Magnetic field: $\vec{H} = \frac{\vec{B}}{\mu} = \frac{\vec{B}}{\mu_0} - \vec{M}$, where $\mu_0 = 4\pi \times 10^{-7}$ (A/m) in the free space.

Magnetic flux density: $\vec{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\vec{\ell} \times \hat{a}_R}{R^2}$

Ampere's law of \vec{H} : $\oint \vec{H} \cdot d\vec{\ell} = I \Leftrightarrow \nabla \times \vec{H} = \vec{J}$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{J} + \vec{J}_m = \vec{J} + \nabla \times \vec{M} \quad \text{or} \quad \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J},$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \oint \vec{H} \cdot d\vec{\ell} = I, \quad \text{and} \quad \vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H},$$

where $\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$

Gauss's law of \vec{B} : $\nabla \cdot \vec{B} = 0 \Leftrightarrow \oiint_S \vec{B} \cdot d\vec{S} = 0$

$\therefore \nabla \cdot \vec{B} = 0$, $\exists \vec{A}$ fulfills $\vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{B} = \mu \vec{J} = \nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Choose $\nabla \cdot \vec{A} = 0 \Rightarrow \nabla^2 \vec{A} = -\mu \vec{J}$ (Note: $\nabla^2 V = -\frac{\rho}{\epsilon}$ is scalar Poisson's equation)

$$\therefore V = \frac{1}{4\pi\epsilon} \iiint_{V'} \frac{\rho}{R} dv', \quad \therefore \vec{A} = \frac{\mu}{4\pi} \iiint_{V'} \frac{\vec{J}}{R} dv' \quad (\text{Wb/m})$$

Magnetic flux: $\Phi = \iint_S \vec{B} \cdot d\vec{S} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{\ell}' \quad (\text{Wb})$

Biot-Savart's law: $\vec{B} = \frac{\mu I}{4\pi} \oint_C \frac{d\vec{\ell}' \times \hat{a}_R}{R^2} = \frac{\mu I}{4\pi} \oint_{C'} \frac{d\vec{\ell}' \times \vec{R}}{R^3}$

$$\vec{A} = \frac{\mu}{4\pi} \iiint_{V'} \frac{\vec{J}}{R} dv' = \frac{\mu I}{4\pi} \oint_{C'} \frac{d\vec{\ell}'}{R}, \quad \therefore \nabla \times (f\vec{G}) = f\nabla \times \vec{G} + (\nabla f) \times \vec{G},$$

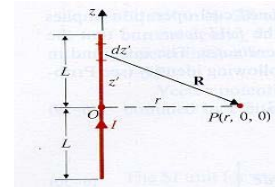
$$\therefore \vec{B} = \nabla \times \vec{A} = \nabla \times \left[\frac{\mu I}{4\pi} \oint_{C'} \frac{d\vec{\ell}'}{R} \right] = \frac{\mu I}{4\pi} \oint_{C'} \nabla \times \left(\frac{d\vec{\ell}'}{R} \right) = \frac{\mu I}{4\pi} \oint_{C'} \left[\frac{1}{R} \nabla \times d\vec{\ell}' + \left(\nabla \frac{1}{R} \right) \times d\vec{\ell}' \right] = \frac{\mu I}{4\pi} \oint_{C'} \frac{d\vec{\ell}' \times \hat{a}_R}{R^2}$$

Note: $\begin{cases} \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \\ d\vec{l}' = \hat{x} dx' + \hat{y} dy' + \hat{z} dz' \Rightarrow \nabla \times d\vec{l}' = 0 \end{cases}$, and then $d\vec{B} = \frac{\mu I}{4\pi} \left(\frac{d\vec{l}' \times \hat{a}_R}{R^2} \right) = \frac{\mu I}{4\pi} \frac{d\vec{l}' \times \vec{R}}{R^3}$

Ex. A direct current I flows in a straight wire of length $2L$. Find the magnetic flux density \vec{B} at a point located at a distance r from the wire in the bisecting plane.

(Sol.) $d\vec{l}' \times \vec{R} = \hat{z} dz' \times (\hat{a}_r r - \hat{z} z') = \hat{a}_\phi r dz'$, $R = (z^2 + r^2)^{1/2}$

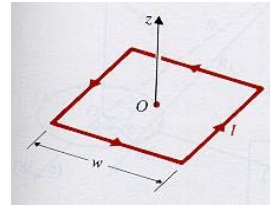
$$\vec{B} = \hat{a}_\phi \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{r dz'}{(z'^2 + r^2)^{3/2}} = \hat{a}_\phi \frac{\mu_0 I L}{2\pi r \sqrt{L^2 + r^2}}$$



Ex. Find the magnetic flux density at the center of a square loop, with side w carrying a direct current I .

(Sol.) $L = \frac{w}{2}$, $r = \frac{w}{2}$ in this case,

$$\vec{B} = 4 \times \hat{z} \frac{\mu_0 I \frac{w}{2}}{2\pi \frac{w}{2} \sqrt{\left(\frac{w}{2}\right)^2 + \left(\frac{w}{2}\right)^2}} = \hat{z} \frac{2\sqrt{2} \mu_0 I}{\pi w}$$

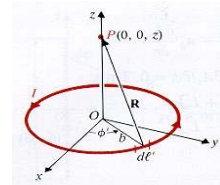


Ex. Find the magnetic flux density at a point on the axis of a circular loop of radius b that a direct current I .

(Sol.) $d\vec{l}' = \hat{a}_\phi b d\phi'$, $\vec{R} = \hat{z} z - \hat{a}_r b$, $R = (z^2 + b^2)^{1/2}$

$$d\vec{l}' \times \vec{R} = \hat{a}_r b z d\phi' + \hat{z} b^2 d\phi'$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{\hat{a}_r b z + \hat{z} b^2}{(z^2 + b^2)^{3/2}} d\phi' = \hat{z} \frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}}$$
. In case of $z=0$, $\vec{B} = \hat{z} \frac{\mu_0 I}{2b}$



Ex. Determine the magnetic flux density at a point on the axis of a solenoid with radius b and length L , and with a current in its N turns of closely wound coil.

(Sol.) $d\vec{B} = \frac{\hat{z} \mu_0 I b^2}{2[(z - z')^2 + b^2]^{3/2}} \left(\frac{N}{L} dz' \right)$

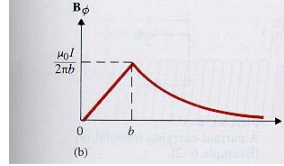
$$\vec{B} = \int_0^L d\vec{B} = \frac{\mu_0 N I}{2L} \left[\frac{L - z}{\sqrt{(L - z)^2 + b^2}} + \frac{z}{\sqrt{z^2 + b^2}} \right] = \frac{\mu_0 N I}{2L} \left[\frac{z}{\sqrt{z^2 + b^2}} - \frac{z - L}{\sqrt{(z - L)^2 + b^2}} \right]$$

Ampere's law of \vec{B} : $\nabla \times \vec{B} = \mu \vec{J} \Leftrightarrow \oint_c \vec{B} \cdot d\vec{l} = \mu I$, where $\mu = \mu_0$ in the free space.

Eg. An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I . Determine the magnetic flux density both inside and outside the conductor. [交大光電所]

(Sol.)

(a) Inside the conductor, $r \leq b$:



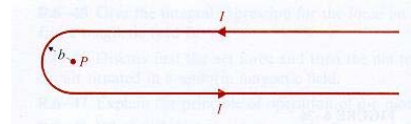
$$\oint_{C_1} \vec{B} \cdot d\vec{l} = \int_0^{2\pi} B r d\phi = 2\pi r B = \mu_0 \left(\frac{\pi r^2}{\pi b^2}\right) I = \mu_0 \left(\frac{r}{b}\right)^2 I \Rightarrow \vec{B} = \hat{a}_\phi \frac{\mu_0 r I}{2\pi b^2}$$

(b) Outside the conductor: $\oint_{C_2} \vec{B} \cdot d\vec{l} = 2\pi r B = \mu_0 I \Rightarrow \vec{B} = \hat{a}_\phi \frac{\mu_0 I}{2\pi r}$

Eg. A long line carrying a current I folds back with semicircular bend of radius b . Determine magnetic flux density at the center point P of the bend. [高考]

(Sol.)

$$\vec{B} = \vec{B}_1 + \vec{B}_2, \text{ where } \vec{B}_1 = 2 \cdot \hat{z} \frac{\mu_0 I}{4\pi b}, \vec{B}_2 = \hat{z} \frac{\mu_0 I}{4b}$$



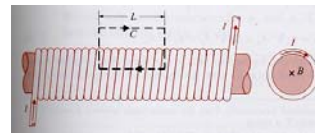
Eg. A current I flows in the inner conductor of an infinitely long coaxial line and returns via the outer conductor. The radius of the inner conductor is a , and the inner and outer radii of the outer conductor are b and c , respectively. Find the magnetic flux density \vec{B} for all regions and plot $|\vec{B}|$ versus r . [高考電機技師]

(Sol.) $0 \leq r \leq a$, $\vec{B} = \hat{a}_\phi \frac{\mu r I}{2\pi a^2}$, $a \leq r \leq b$, $\vec{B} = \hat{a}_\phi \frac{\mu I}{2\pi r}$

$$b \leq r \leq c, \vec{B} = \hat{a}_\phi \left(\frac{c^2 - r^2}{c^2 - b^2}\right) \frac{\mu I}{2\pi r}$$

Eg. Determine the magnetic flux density inside an infinitely long solenoid with air core having n closely wound turns per unit length and carrying a current I .

(Sol.) $BL = \mu_0 n L I \Rightarrow B = \mu_0 n I$



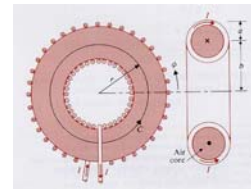
Eg. The figure shows an infinitely long solenoid with air core having n closely wound turns per unit length. The windings are slanted at an angle α and carry a current I . Determine the magnetic flux density both inside and outside the solenoid.



$$\text{(Sol.) } \vec{B}_1 = \begin{cases} 0, & 0 < r < b \\ \hat{a}_\phi \frac{2\pi\mu_0 bnI \sin\alpha}{2\pi r} = \hat{a}_\phi \frac{\mu_0 bnI \sin\alpha}{r}, & r > b \end{cases}$$

$$\vec{B}_2 = \begin{cases} \hat{z}\mu_0 nI \cos\alpha, & 0 < r < b \\ 0, & r > b \end{cases}, \quad \vec{B} = \vec{B}_1 + \vec{B}_2$$

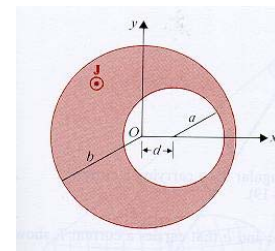
Eg. Determine the magnetic flux density inside a closely wound toroidal coil with an air core having N turns and carrying a current I . The toroid has a mean radius b , and the radius of each turn is a .



$$\text{(Sol.) } \oint \vec{B} \cdot d\vec{l} = 2\pi r B = \mu_0 NI$$

$$\vec{B} = \hat{a}_\phi B = \hat{a}_\phi \frac{\mu_0 NI}{2\pi r}, \quad (b-a) < r < (b+a), \quad \vec{B} = 0 \quad \text{for } r < (b-a) \text{ and } r > (b+a)$$

Eg. In certain experiments it is desirable to have a region of constant magnetic flux density. This can be created in an off-center cylindrical cavity. The uniform axial current density is $\vec{J} = \hat{z}J$. Find the magnitude and direction of \vec{B} in the cylindrical cavity whose axis is displaced from that of the conducting part by a distance d . [台大電研、清大電研、中原電機]



$$\text{(Sol.) } \vec{J} = \hat{z}J, \quad \oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

If no hole exists,

$$2\pi r_1 B_{\phi 1} = \mu_0 \pi r_1^2 J \Rightarrow B_{\phi 1} = \frac{\mu_0 r_1}{2} J \Rightarrow \begin{cases} B_{x1} = -\frac{\mu_0 J}{2} y_1 \\ B_{y1} = \frac{\mu_0 J}{2} x_1 \end{cases}$$

$$\text{For } -\vec{J} \text{ in the hole portion, } B_{\phi 2} = -\frac{\mu_0 r_2}{2} J \Rightarrow \begin{cases} B_{x2} = \frac{\mu_0 J}{2} y_2 \\ B_{y2} = -\frac{\mu_0 J}{2} x_2 \end{cases}$$

$$\text{At } y_1 = y_2 \text{ and } x_1 = x_2 + d \Rightarrow B_x = B_{x1} + B_{x2} = 0, \text{ and } B_y = B_{y1} + B_{y2} = \frac{\mu_0 J}{2} d$$

1-4 Electromagnetic Forces

Lorentz force equation: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Electric force: $\vec{F}_e = q\vec{E}$. **Magnetic force:** $\vec{F}_m = q\vec{v} \times \vec{B}$

Ex. An electron is injected with an initial velocity $\vec{v}_0 = \hat{y}v_0$ into a region where both an electric field and a magnetic field \vec{B} exist. Describe the motion of the electron if $\vec{E} = \hat{z}E_0$ and $\vec{B} = \hat{x}B_0$. Discuss the effect of the relative magnitude of E_0 and B_0 on the electron paths in parts.

(Sol.) $m \frac{\partial \vec{v}}{\partial t} = -e(\vec{E} + \vec{v} \times \vec{B}),$

$$\begin{cases} \vec{E} = \hat{z}E_0 \\ \vec{B} = \hat{x}B_0 \end{cases} \Rightarrow \begin{cases} \frac{\partial v_x}{\partial t} = 0 \\ \frac{\partial v_y}{\partial t} = -\frac{e}{m} B_0 v_z \\ \frac{\partial v_z}{\partial t} = -\frac{e}{m} (E_0 - B_0 v_y) \end{cases} \Rightarrow \begin{cases} v_x = 0 \\ v_y = (v_0 - \frac{E_0}{B_0}) \cos \omega_0 t + \frac{E_0}{B_0} \\ v_z = (\frac{E_0}{B_0} - v_0) \sin \omega_0 t; \omega_0 = \frac{e}{m} B_0 \end{cases}$$

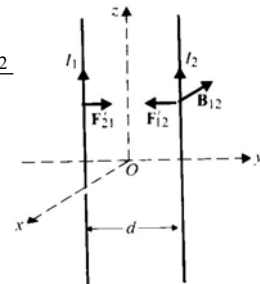
$$\text{(If } v_0 \neq \frac{E_0}{B_0} \text{)} \Rightarrow \begin{cases} x = 0 \\ y = \frac{c_2}{\omega_0} \sin \omega_0 t + \frac{E_0}{B_0} t \\ z = -\frac{c_2}{\omega_0} (1 - \cos \omega_0 t), \quad c_2 = v_0 - \frac{E_0}{B_0} \end{cases} \Rightarrow (y - \frac{E_0}{B_0} t)^2 + (z + \frac{c_2}{\omega_0})^2 = (\frac{c_2}{\omega_0})^2$$

Magnetic force due to \vec{B} and I :

$$F_m = q\vec{v} \times \vec{B} \Rightarrow dF_m = dq \frac{d\vec{\ell}}{dt} \times \vec{B} = \frac{dq}{dt} d\vec{\ell} \times \vec{B} = Id\vec{\ell} \times \vec{B}, \quad \therefore F_m = I \oint_C d\vec{\ell} \times \vec{B}$$

Ex. Determine the force per unit length between two infinitely long parallel conducting wires carrying currents I_1 and I_2 in the same direction. The wires are separated by a distance d . [清大電研]

(Sol.) $\vec{F}_{12}' = I_2(\hat{z} \times \vec{B}_{12}), \quad \vec{B}_{12} = -\hat{x} \frac{\mu_0 I_1}{2\pi d} \Rightarrow \vec{F}_{12}' = -\hat{y} \frac{\mu_0 I_1 I_2}{2\pi d}$

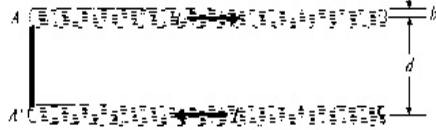


Eg. Calculate the force per unit length on each of three equidistant, infinitely long, parallel wires d apart, each carrying a current of I in the same direction. Specify the direction of the force.

(Sol.) $I_1 = I_2 = I_3 = I$, $\vec{B}_2 = \hat{x}2B_{12} \cos 30^\circ = \hat{x} \frac{\sqrt{3}\mu_0 I}{2\pi d}$, $\vec{f}_2 = -\hat{z}I \times \vec{B}_2 = -\hat{y}IB_2 = -\hat{y} \frac{\sqrt{3}\mu_0 I^2}{2\pi d}$

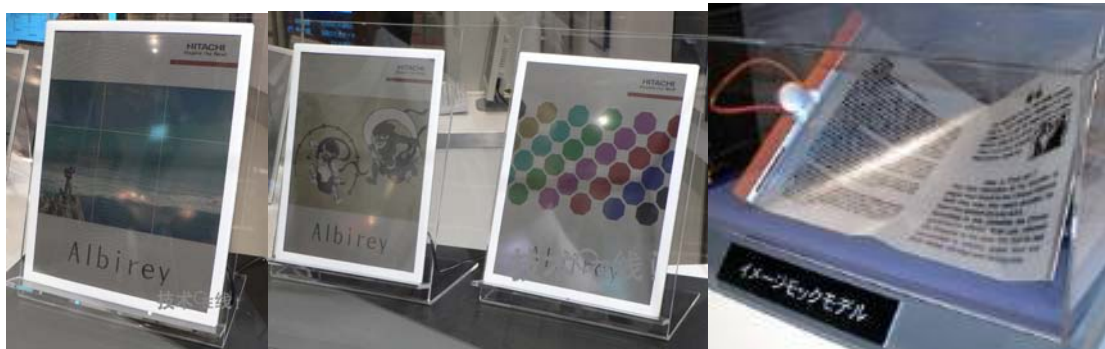
Eg. The bar AA', serves as a conducting path for the current I in two very long parallel lines. The lines have a radius b and are spaced at a distance d apart. Find the direction and the magnitude of the magnetic force on the bar. [中山物理所]

(Sol.) $\vec{B} = -\hat{z} \frac{\mu_0 I}{4\pi} \left(\frac{1}{y} + \frac{1}{d-y} \right)$, $d\vec{\ell} = \hat{y}dy$

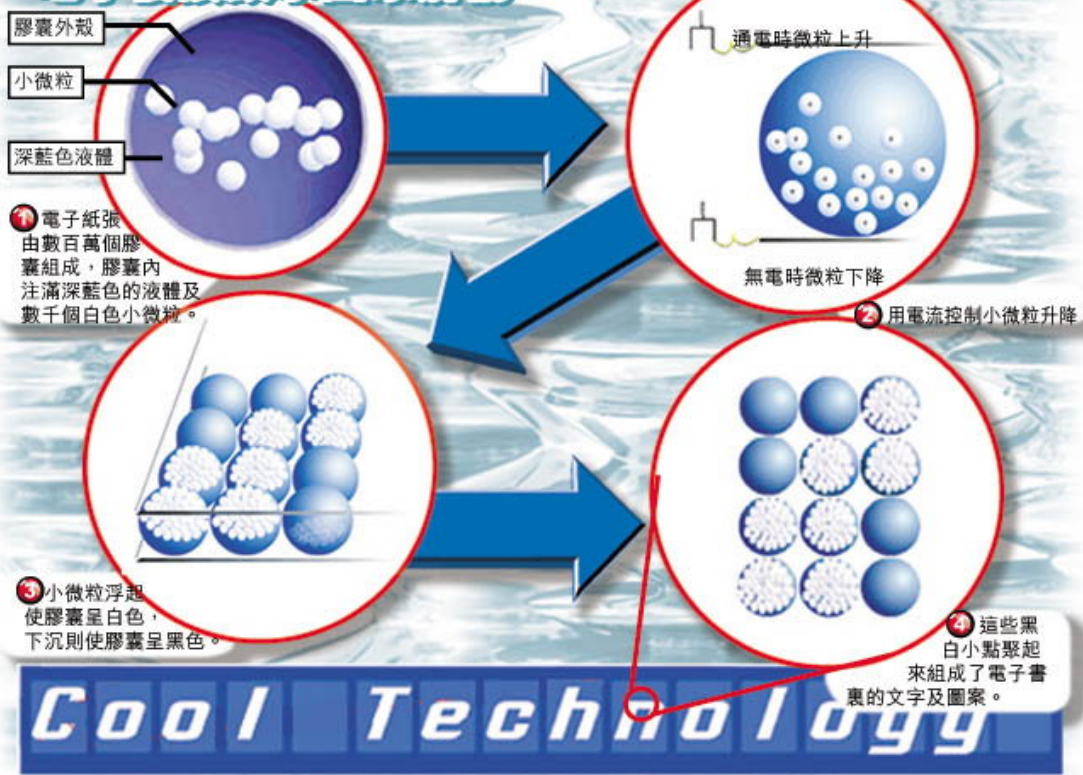


$\Rightarrow d\vec{F} = Id\vec{\ell} \times \vec{B} = -\hat{x} \frac{\mu_0 I^2}{4\pi} \left(\frac{1}{y} + \frac{1}{d-y} \right) dy \Rightarrow \vec{F} = \int_b^{d-b} d\vec{F} = -\hat{x} \frac{\mu_0 I^2}{2\pi} \ln\left(\frac{d}{b} - 1\right)$

Application of the electric forces: The e-paper



電子紙張顯示圖象解構



Scalar electric potential function: $V = \frac{1}{4\pi\epsilon} \iiint_{V'} \frac{\rho}{R} dv'$

Vector magnetic potential function: $\vec{A} = \frac{\mu}{4\pi} \iiint_{V'} \frac{\vec{J}}{R} dv'$

Retarded potentials:

$$V(R,t) = \frac{1}{4\pi\epsilon} \iiint_{V'} \frac{\rho(t-R/v)}{R} dv', \quad \vec{A}(R,t) = \frac{\mu}{4\pi} \iiint_{V'} \frac{\vec{J}(t-R/v)}{R} dv'$$

1-5 Faraday's Law and Magnetic Dipoles

Faraday's law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ or $\oint_C \vec{E} \cdot d\vec{\ell} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$

$$\therefore \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) \Rightarrow \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$$

$$\therefore \exists V \text{ fulfills } \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \Rightarrow \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Note: In static field: $\vec{E} = -\nabla V$, but in time-varying field: $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$

Emf: $V = \oint_C \vec{E} \cdot d\vec{\ell}$. **Magnetic flux:** $\Phi = \iint_S \vec{B} \cdot d\vec{S}$, $\oint_C \vec{E} \cdot d\vec{\ell} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \Rightarrow V = -\frac{d\Phi}{dt}$

Motional emf: $V' = \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{\ell}$ (Volt)

$$\vec{F}_m = q\vec{v} \times \vec{B} \Rightarrow \frac{\vec{F}_m}{q} = \vec{v} \times \vec{B} \equiv -\vec{E}_m \Rightarrow V' = -\oint_C \vec{E}_m \cdot d\vec{\ell} = \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{\ell}$$

Ex. A circular loop of N turns of conducting wire line in the xy -plane with its center at the origin of a magnetic field specified by $\vec{B} = \hat{z}B_0 \cos(\pi r/2b) \sin \omega t$, where b is the radius of the loop and ω is the angular frequency. Find the *emf* induced in the loop.

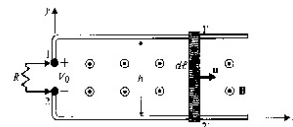
(Sol.) $\Phi = \int_0^b [\hat{z}B_0 \cos(\frac{\pi r}{2b}) \sin(\omega t)] \cdot [\hat{z}2\pi r dr] = \frac{8b^2}{\pi} (\frac{\pi}{2} - 1) B_0 \sin(\omega t)$

$$V = -\frac{Nd\Phi}{dt} = -\frac{8N}{\pi} b^2 (\frac{\pi}{2} - 1) B_0 \omega \cos(\omega t)$$

Ex. A metal bar slides over a pair of conducting rails in a uniform magnetic field $\vec{B} = \hat{z}B_0$ with a constant velocity v . (a) Determine the open-circuit voltage V_0 that appears across terminals 1 and 2. (b) Assuming that a resistance R is connected between the terminals, find the electric power dissipated in R . Neglect the electric resistance of the metal bar and of the conducting rails. [交大電子物理所]

(Sol.) (a) $V_0 = V_1 - V_2 = \int_2^1 (\hat{x}v \times \hat{z}B_0) \cdot (\hat{y}dl) = -vB_0h$ (V)

(b) $P_e = I^2 R = (\frac{vB_0h}{R})^2 R = \frac{(vB_0h)^2}{R}$ (W)



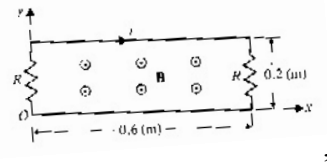
Eg. The circuit in Fig. is situated in a magnetic field $\vec{B} = \hat{z}3 \cos(5\pi 10^7 t - \frac{2}{3}\pi x)$

μT . Assuming $R=15\Omega$, find the current i . [中山物理所]

(Sol.) $\Phi = \int_0^{0.6} 3 \cos(5\pi \times 10^7 t - \frac{2}{3}\pi x) 10^{-6} \cdot (0.2 dx)$

$V = -\frac{d\Phi}{dt} = 45[\cos(5\pi \times 10^7 t - \frac{2}{3}\pi 0.6) - \cos(5\pi 10^9 t)]$

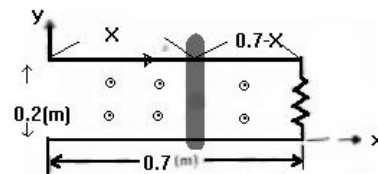
$i = \frac{V}{2R} = 1.76 \sin(5\pi 10^7 t - 0.2\pi)$



Eg. A conducting sliding bar oscillates over two parallel conducting rails in a sinusoidally varying magnetic field $\vec{B} = \hat{z}5 \cos(\omega t)$ T. The position of the sliding bar is given by $x=0.35(1-\cos\omega t)$, and the rails are terminated in a resistance $R=0.2\Omega$. Find i .

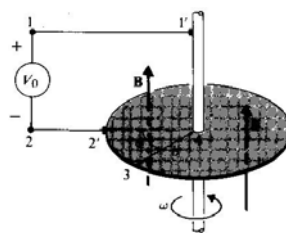
(Sol.) $\Phi = 5 \cos \omega t \cdot 0.2(0.7 - x)$, $x=0.35(1-\cos\omega t)$, $i = -\frac{1}{R} \frac{d\Phi}{dt}$

$\Rightarrow i = 1.75\omega \sin \omega t \cdot (1 + 2 \cos \omega t)$



Eg. The Faraday disk generator consists of a circular metal disk rotating with a constant angular velocity ω in a uniform and constant magnetic field of flux density $B = \hat{z}B_0$ that is parallel to the axis of rotation. Brush contacts are the open-circuit voltage of the generator if the radius of the disk is b .

(Sol.) $V_0 = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = \int_3^4 [(\hat{a}_\phi r \omega) \times \hat{z}B_0] \cdot (\hat{a}_r dr) = \omega B_0 \int_b^0 r dr = \frac{\omega B_0 b^2}{2}$ (V)



Eg. A rectangular loop of width w and height h is situated near a very long wire carrying a current i_1 . Assume i_1 to be a rectangular pulse. Find the induced current i_2 in the rectangular loop whose self-inductance is L .

(Sol.)

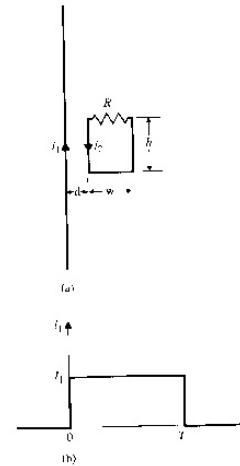
$$L_{12} \frac{di_1}{dt} = L \frac{di_2}{dt} + Ri_2,$$

$$\text{where } L_{12} = \frac{\Phi_{12}}{i_1} = \frac{h}{i_1} \int_d^{d+w} \frac{\mu_0 i_1}{2\pi r} dr = \frac{\mu_0 h}{2\pi} \ln\left(1 + \frac{w}{d}\right)$$

$$t=0, \quad L \frac{di_2}{dt} + Ri_2 = L_{12} I_1 \delta(t) \Rightarrow i_2 = \frac{L_{12}}{L} I_1 e^{-\left(\frac{R}{L}\right)t}$$

$$t=T, \quad i_2 = \frac{L_{12}}{L} I_1 e^{-\frac{RT}{L}}, \quad \text{when } -I_1 \text{ is applied}$$

$$t>T, \quad i_2 = -\frac{L_{12}}{L} I_1 e^{-\left(\frac{R}{L}\right)(t-T)}$$



Magnetic dipole moment: $\vec{m} = \hat{z}IS = \hat{z}m$, where S is the area of the loop that carries I .

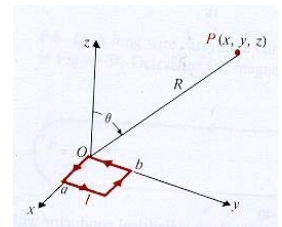
Vector potential of a magnetic dipole: $\vec{A} = \frac{\mu_0 m \times \hat{a}_R}{4\pi R^2}$, where $m=IS$ and

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 m}{4\pi R^3} (\hat{a}_R 2 \cos \theta + \hat{a}_\theta \sin \theta)$$

Eg. For the small rectangular loop with sides a and b that carries a current I . Find the vector magnetic potential \vec{A} at a distant point $P(x,y,z)$. And determine the magnetic flux density \vec{B} and \vec{A} . [交大光電所]

(Sol.)

$$\vec{A} = \frac{\mu_0 \vec{m} \times \hat{a}_R}{4\pi R^2}, \quad \text{where } m=Iab, \quad \vec{B} = \nabla \times \vec{A} = \frac{\mu_0 m}{4\pi R^3} (\hat{a}_R 2 \cos \theta + \hat{a}_\theta \sin \theta)$$



Magnetization vector: $\vec{M} = \lim_{\Delta V \rightarrow 0} \frac{\sum_{k=1}^{\Delta V} \vec{m}_k}{\Delta V}$ (A/m), where \vec{m}_k is the magnetic dipole moment of an atom.

$$d\vec{A} = \frac{\mu_0 \vec{M} \times \hat{a}_R}{4\pi R^2} dv' = \frac{\mu_0}{4\pi} \vec{M} \times \nabla' \left(\frac{1}{R} \right) dv' = \frac{\mu_0}{4\pi} \left[\frac{1}{R} \nabla' \times \vec{M} - \nabla' \times \left(\frac{\vec{M}}{R} \right) \right]$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \iiint_{V'} \frac{\nabla' \times \vec{M}}{R} dv' - \frac{\mu_0}{4\pi} \iiint_{V'} \nabla' \times \left(\frac{\vec{M}}{R} \right) dv' \quad \left(\iiint_{V'} \nabla' \times \vec{F} dv' = - \oiint_S \vec{F} \times dS' \right)$$

$$= \frac{\mu_0}{4\pi} \iiint_{V'} \frac{\nabla' \times \vec{M}}{R} dv' + \frac{\mu_0}{4\pi} \oiint_S \frac{\vec{M} \times \hat{a}_n'}{R} dS'$$

\therefore Magnetization volume current density: $\boxed{\vec{J}_m = \nabla \times \vec{M}}$ (A/m²)

Magnetization surface current density: $\boxed{\vec{J}_{ms} = \vec{M} \times \hat{a}_n}$ (A/m)

Equivalent Magnetization Charge Densities:

$$V_m = \frac{1}{4\pi} \oiint_S \frac{\vec{M} \cdot \hat{a}_n'}{R} dS' + \frac{1}{4\pi} \iiint_{V'} -\frac{(\nabla' \cdot \vec{M})}{R} dv' \quad (\text{Note: } V = \frac{1}{4\pi\epsilon_0} \oiint_S \frac{\vec{P} \cdot \hat{a}_n'}{R} dS' + \frac{1}{4\pi\epsilon_0} \iiint_{V'} -\frac{(\nabla' \cdot \vec{P})}{R} dv')$$

Define the magnetization surface charge density as $\boxed{\rho_{ms} = \vec{M} \cdot \hat{a}_n}$ and the

magnetization volume charge density as $\boxed{\rho_m = -\nabla \cdot \vec{M}}$

Ex. A circular rod of magnetic material with permeability μ is inserted coaxially in the long solenoid. The radius a of the rod is less than the inner radius b of the solenoid. The solenoid's winding has n turns per unit length and carries a current I . (a) Find the values of \vec{B} , \vec{H} , and \vec{M} inside the solenoid for $r < a$ and for $a < r < b$. (b) What are the equivalent magnetization current densities J_m and J_{ms} for the magnetized rod? [清大電研]

$$(\text{Sol.}) \text{ (a) } r < a: \vec{H} = \hat{z}nI, \vec{B} = \hat{z}\mu nI, \vec{M} = \frac{\vec{B}}{\mu_0} - \vec{H} = \hat{z} \left(\frac{\mu}{\mu_0} - 1 \right) nI,$$

$$a < r < b: \vec{H} = \hat{z}nI, \vec{B} = \hat{z}\mu_0 nI, \vec{M} = 0$$

$$\text{(b) } \vec{J}_m = \nabla' \times \vec{M} = 0, \vec{J}_{ms} = \vec{M} \times \hat{a}_n = (\hat{z} \times \hat{a}_r) \left(\frac{\mu}{\mu_0} - 1 \right) nI = \hat{a}_\phi \left(\frac{\mu}{\mu_0} - 1 \right) nI$$

Eg. A ferromagnetic sphere of radius b is uniformly magnetized with a magnetization $\vec{M} = \hat{z}M_0$. (a) Determine the equivalent magnetization current densities \vec{J}_m and \vec{J}_{ms} . (b) Determine the magnetic flux density at the center of the sphere. [台大電研]

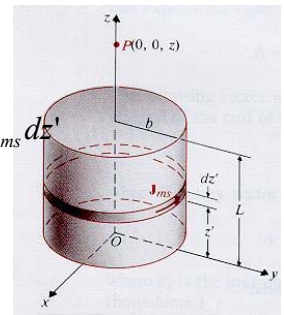
(Sol.) (a) $\vec{J}_m = \nabla' \times \vec{M} = 0$, $\vec{J}_{ms} = (\hat{a}_R \cos \theta - \hat{a}_\theta \sin \theta)M_0 \times \hat{a}_R = \hat{a}_\phi M_0 \sin \theta$

(b) $d\vec{B} = \hat{z} \frac{\mu_0 (J_{ms} b d\theta)(b \sin \theta)^2}{2(b^2)^{3/2}} = \hat{z} \frac{\mu_0 M_0}{2} \sin^3 \theta$, $\vec{B} = \hat{z} \frac{\mu_0 M_0}{2} \int_0^\pi \sin^3 \theta d\theta = \hat{z} \frac{2}{3} \mu_0 M_0$.

Eg. Determine the magnetic flux density on the axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius b , length L and axial magnetization $\vec{M} = \hat{z}M_0$. [台大物研]

(Sol.) $\vec{J}_m = \nabla' \times \vec{M} = 0$, $\vec{J}_{ms} = \vec{M} \times \hat{a}_n = (\hat{z}M_0) \times \hat{a}_r = \hat{a}_\phi M_0$, $dI = J_{ms} dz'$

$$\vec{B} = \hat{z} \int_0^L \frac{\mu_0 M_0 b^2 dz'}{2[(z-z')^2 + b^2]^{3/2}} = \hat{z} \frac{\mu_0 M_0}{2} \left[\frac{z}{\sqrt{z^2 + b^2}} - \frac{z-L}{\sqrt{(z-L)^2 + b^2}} \right]$$



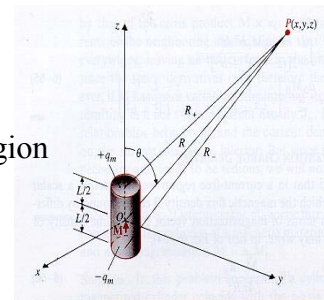
Eg. A cylindrical bar magnet of radius b and length L has a uniform magnetization $\vec{M} = \hat{z}M_0$ along its axis. Use the equivalent magnetization charge density concept to determine the magnetic flux density at an arbitrary distant point. [交大電子所]

(Sol.) $\rho_{ms} = \vec{M} \cdot \hat{a}_n = \begin{cases} M_0, & \text{top} \\ -M_0, & \text{bottom}, \rho_m=0 \text{ in the interior region} \\ 0, & \text{sidewall} \end{cases}$

$$q_m = \pi b^2 \rho_{ms} = \pi b^2 M_0 \Rightarrow V_m(x, y, z) = \frac{q_m}{4\pi} \left(\frac{1}{R_+} - \frac{1}{R_-} \right) \quad (A)$$

$$= \frac{q_m L \cos \theta}{4\pi R^2} = \frac{(\pi b^2 M_0) L \cos \theta}{4\pi R^2} = \frac{M_T \cos \theta}{4\pi R^2}, \text{ where } M_T = \pi b^2 L M_0$$

$$\vec{B} = -\mu_0 \nabla V_m = \frac{\mu_0 M_T}{4\pi R^3} (\hat{a}_R 2 \cos \theta + \hat{a}_\theta \sin \theta) \quad (T)$$



Consider an infinitely long solenoid with n turns per unit length around to create a magnetic field; a voltage $V_1 = -n \frac{d\Phi}{dt}$ is induced unit length, which opposes the current change. Power $P_1 = -V_1 I$ per unit length must be supplied to overcome this induced voltage in order to increase the current to I . The work per unit volume required to produce a final magnetic flux density B_f is $W_1 = \int_0^{B_f} H dB$.