

## Chapter 4 Selected Topics for Circuits and Systems

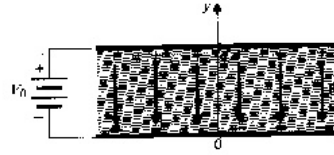
### 4-1 Poisson's and Laplace's Equations

**Poisson's equation:**  $\nabla \cdot \epsilon \vec{E} = \rho \Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon}$

**Laplace's equation:** If no charge exists,  $\rho=0$ ,  $\nabla^2 V = 0$

**Eg.** The two plates of a parallel-plate capacitor are separated by a distance  $d$  and maintained at potentials 0 and  $V_0$ . Assume negligible fringing effect at the edges, determine (a) the potential at any point between the plates, (b) the surface charge densities on the plates. [清大電研]

(Sol.)  $\frac{d^2 V}{dy^2} = 0 \Rightarrow V = c_1 y + c_2$ ,  $V(y=0) = 0$ ,  $V(y=d) = V_0 \Rightarrow V = \frac{V_0}{d} y$



$$\vec{E} = -\hat{y} \frac{dV}{dy} = -\hat{y} \frac{V_0}{d}, E_n = \hat{a}_n \cdot \vec{E} = \frac{\rho}{\epsilon}$$

At the lower plate:  $\hat{a}_n = \hat{y}$ ,  $\rho_{sl} = -\frac{\epsilon V_0}{d}$ .

At the upper plate:  $\hat{a}_n = -\hat{y}$ ,  $\rho_{ul} = \frac{\epsilon V_0}{d}$

**Eg.** The upper and lower conducting plates of a large parallel-plate capacitor are separated by a distance  $d$  and maintained at potentials  $V_0$  and 0, respectively. A dielectric slab of dielectric constant  $\epsilon_r$  and uniform thickness  $0.8d$  is placed over the lower plate. Assuming negligible fringing effect, determine (a) the potential and electric field distribution in the dielectric slab, (b) the potential and electric field distribution in the air space between the dielectric slab and the upper plate. [台大電研]

(Sol.) Set  $V_d(y) = c_1 y + c_2$ ,  $\vec{E}_d = -\hat{y} c_1$ ,  $D_d = -\hat{y} \epsilon_0 \epsilon_r c_1$

$$V_a(y) = c_3 y + c_4, \vec{E}_a = -\hat{y} c_3, D_a = -\hat{y} \epsilon_0 c_3$$

$$V_d(0) = 0, V_a(d) = V_0, V_d(0.8d) = V_a(0.8d), D_d(0.8d) = D_a(0.8d)$$

$$\Rightarrow c_1 = \frac{V_0}{(0.8 + 0.2\epsilon_r)d}, c_2 = 0, c_3 = \frac{\epsilon_r V_0}{(0.8 + 0.2\epsilon_r)d}, c_4 = \frac{(1 - \epsilon_r)V_0}{1 + 0.25\epsilon_r}$$

$$(a) V_d = \frac{5yV_0}{(4 + \epsilon_r)d}, \vec{E}_d = -\hat{y} \frac{5V_0}{(4 + \epsilon_r)d} \quad (b) V_a = \frac{5\epsilon_r y - 4(\epsilon_r - 1)d}{(4 + \epsilon_r)d} V_0, \vec{E}_a = -\hat{y} \frac{5\epsilon_r V_0}{(4 + \epsilon_r)d}$$

**Eg. Show that uniqueness of electrostatic solutions.**

(Proof) Let  $V_1$  and  $V_2$  satisfy  $\nabla^2 V_1 = -\frac{\rho}{\epsilon}$  and  $\nabla^2 V_2 = -\frac{\rho}{\epsilon}$ . Define  $V_d = V_1 - V_2$ ,

$$\nabla^2 V_d = 0$$

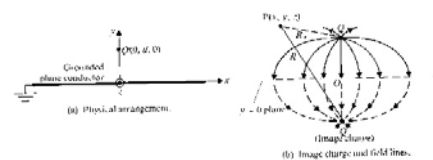
1. On the conducting boundaries,  $V_d = 0 \Rightarrow V_1 = V_2$

2. Let  $f = V_d$ ,  $\vec{A} = \nabla V_d$

$$\nabla \cdot (V_d \nabla V_d) = \nabla \cdot (f \vec{A}) = f \nabla \cdot \vec{A} + \vec{A} \cdot \nabla f = V_d \nabla^2 V_d + |\nabla V_d|^2 \Rightarrow \iint_s (V_d \nabla V_d) \cdot \hat{a}_n ds = \iiint_v |\nabla V_d|^2 dv$$

$$R \rightarrow \infty, V_d = V_1 - V_2 \propto \frac{1}{R}, \nabla V_d \propto \frac{1}{R^2}, ds \propto R^2 \Rightarrow \iint_s (V_d \nabla V_d) \cdot \hat{a}_n ds \rightarrow 0, \therefore \iiint_v |\nabla V_d|^2 dv = 0 \Rightarrow$$

$$V_d = 0 \Rightarrow V_1 = V_2$$



**Image Theorem**  $P(x,y,z)$  in the  $y>0$  region is

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{R_+} - \frac{1}{R_-} \right), \text{ where } R_+ \text{ and } R_- \text{ are}$$

the distances from  $Q$  and  $-Q$  to the point  $P$ ,

respectively.

**Eg. A point charge  $Q$  exists at a distance  $d$  above a large grounded conducting plane. Determine (a) the surface charge density  $\rho_s$ , (b) the total charge induced on the conducting plane. [交大光電所]**

$$\text{(Sol.) } \vec{E}_{/y=0} = -\hat{y} \frac{Q}{4\pi\epsilon_0 R^2} \cdot 2 \sin \theta = -\hat{y} \frac{Qd}{2\pi\epsilon_0 (d^2 + r^2)^{3/2}}$$

$$\text{(a) } \rho_s = \hat{y} \cdot \epsilon \vec{E}_{/y=0} = -\frac{Qd}{2\pi(d^2 + r^2)^{3/2}}, \text{ (b) } \int_0^\infty \rho_s 2\pi r dr = -Q$$

**Eg. Two dielectric media with dielectric constants  $\epsilon_1$  and  $\epsilon_2$  are separated by a plane boundary at  $x=0$ . A point charge  $Q$  exists in medium 1 at distance  $d$  from the boundary. Determine  $Q_1$  &  $Q_2$ . ( $Q$  &  $-Q_1$  in medium 1, or  $Q$  &  $Q_2$  in medium 2)**

$$\text{(Sol.) } V_1(x=0) = \frac{Q}{4\pi\epsilon_1 \sqrt{s^2 + d^2}} - \frac{Q_1}{4\pi\epsilon_1 \sqrt{s^2 + d^2}}, \quad V_2(x=0) = \frac{Q + Q_2}{4\pi\epsilon_2 \sqrt{s^2 + d^2}}$$

$$V_1 = V_2, \quad \epsilon_1 \frac{\partial V_1}{\partial x} = \epsilon_2 \frac{\partial V_2}{\partial x} \text{ at } x=0 \Rightarrow \frac{Q - Q_1}{\epsilon_1} = \frac{Q + Q_2}{\epsilon_2}, \text{ and}$$

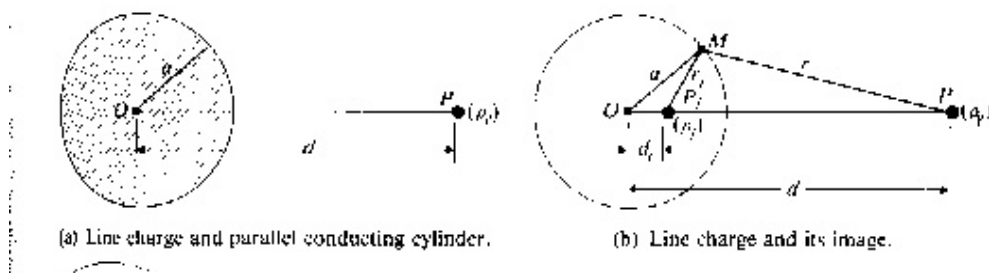
$$Q + Q_1 = Q + Q_2 \Rightarrow Q_1 = Q_2 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} Q$$

**Ex. A line charge density  $\rho_l$  located at a distance  $d$  from the axis of a parallel conducting circular cylinder of radius  $a$ . Both are infinitely long. Find the image position of line charge.**

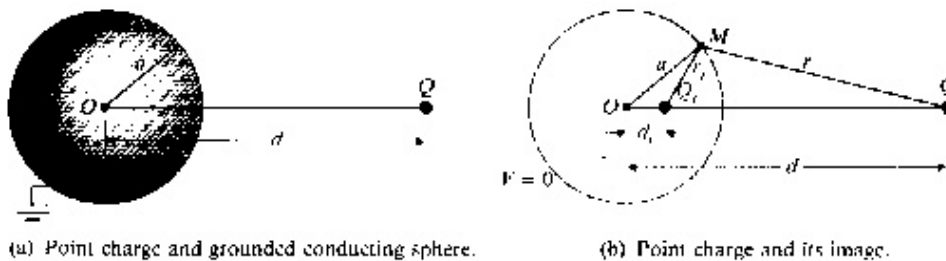
(Sol.) Assume  $\rho_i = -\rho_l, V = -\int_{r_0}^r E dr = -\frac{\rho_l}{2\pi\epsilon_0} \int_{r_0}^r \frac{1}{r} dr = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_0}{r}$

$\Rightarrow V_M = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_0}{r} - \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_0}{r_i} = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{r_i}{r}$

$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = C \Rightarrow d_i = \frac{a^2}{d}$



**Ex. A point charge  $Q$  is placed at a distance  $d$  to a conducting sphere. Find its image.**



(Sol.)  $V_M = \frac{1}{4\pi\epsilon_0} \left( \frac{Q}{r} + \frac{Q_i}{r_i} \right) = 0 \Rightarrow \frac{r_i}{r} = -\frac{Q_i}{Q} = \frac{a}{d} \Rightarrow Q_i = -\frac{a}{d} Q, \frac{a-d_i}{d-a} = \frac{a}{d} \Rightarrow d_i = \frac{a^2}{d}$

## 4-2 Boundary-Value Problems in Rectangular Coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \text{ Let } V(x,y,z) = X(x)Y(y)Z(z), k_x^2 + k_y^2 + k_z^2 = 0$$

$$\rightarrow \frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0, \frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0, \frac{d^2 Z(z)}{dz^2} + k_z^2 Z(z) = 0$$

For  $X(x)$ , 1.  $k_x^2 = 0$ ,  $X(x) = A_0 x + B_0$  is linear.

2.  $k_x^2 > 0$ ,  $X(x) = A_1 \sin k_x x + B_1 \cos k_x x$ ,  $X(x=a)$  is finite,  $X(x=b)$  is finite

3.  $k_x^2 < 0$ ,  $X(x) = A_2 \sinh k_x x + B_2 \cosh k_x x$ ,  $X(\infty)$  is finite,  $X(-\infty)$  is finite

Similar cases exist in  $Y(y)$  and  $Z(z)$ .

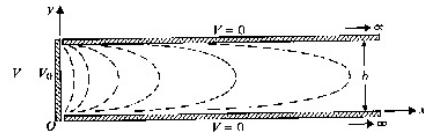
**Eg. Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance  $b$ . A third electrode perpendicular to and insulated from both is maintained at a constant potential  $V_0$ . Determine the potential distribution in the region enclosed by the electrodes. [高考]**

(Sol.)

$$V(x, y, z) = V(x, y) = X(x)Y(y)$$

$$B.C.: V(0, y) = V_0, V(x, 0) = 0$$

$$V(\infty, y) = 0, V(x, b) = 0$$



$$\frac{d^2 X(x)}{dx^2} = k_x^2 X(x) \Rightarrow X(x) = D_1 e^{k_x x} + D_2 e^{-k_x x} = D_2 e^{-k_x x}$$

$$\frac{d^2 Y(y)}{dy^2} = -k_y^2 Y(y) \Rightarrow Y(y) = A_1 \sin k_y y$$

$$\Rightarrow V_n(x, y) = C_n e^{-k_x x} \sin k_y y \Rightarrow k_y = \frac{n\pi}{b}, n = 1, 2, 3$$

$$\Rightarrow V_n(x, y) = C_n e^{-\frac{n\pi x}{b}} \sin \frac{n\pi y}{b}$$

$$V(0, y) = V_0 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{b}$$

$$\int_0^b V_0 \sin \frac{m\pi y}{b} dy = \sum_{n=1}^{\infty} C_n \int_0^b \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b} dy$$

$$= \begin{cases} \frac{2bV_0}{m\pi}, m : \text{odd} \\ 0, m : \text{even} \end{cases} = \begin{cases} \frac{C_n}{2}, m = n \\ 0, m \neq n \end{cases}$$

$$\Rightarrow C_n = \begin{cases} \frac{4V_0}{n\pi}, n : \text{odd} \\ 0, n : \text{even} \end{cases}$$

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n:\text{odd}} \frac{1}{n} e^{-\frac{n\pi x}{b}} \sin \frac{n\pi y}{b}, n = 1, 3, 5 \text{ for } x > 0, 0 < y < b.$$

### 4-3 Boundary-Value Problems in Cylindrical Coordinates

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

1. Assume  $\frac{\partial^2 V}{\partial z^2} = 0$ , then  $V(r, \phi) = R(r)\Phi(\phi)$ ,

$$\rightarrow r^2 \frac{d^2 R(r)}{dr^2} + r \frac{dR(r)}{dr} - n^2 R(r) = 0, \quad \frac{d^2 \Phi(\phi)}{d\phi^2} + n^2 \Phi(\phi) = 0$$

$$\rightarrow R(r) = A_1 r^n + B_1 r^{-n}, \quad \Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi$$

$$\rightarrow V_n(r, \phi) = r^n (A \sin n\phi + B \cos n\phi) + r^{-n} (A' \sin n\phi + B' \cos n\phi), \quad n \neq 0$$

$$\rightarrow V(r, \phi) = \sum_{n=1}^{\infty} V_n(r, \phi)$$

2. Assume  $n=0$ ,  $\frac{d^2 \Phi(\phi)}{d\phi^2} = 0 \Rightarrow \Phi(\phi) = A_0 \phi + B_0$ ,  $R(r) = C_0 \ln r + D_0$ ,

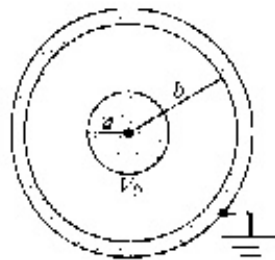
In the  $\phi$ -independent case,  $V(r) = C_1 \ln r + D_1$

In the  $\phi$ -dependent case,  $\Phi(\phi) = A\phi + B$ ,  $V(r, \phi) = (C \ln r + D)(A\phi + B)$

**Eg. Consider a very long coaxial cable. The inner conductor has a radius  $a$  and is maintained at a potential  $V_0$ . The outer conductor has an inner radius  $b$  and is grounded. Determine the potential distribution in the space between the conductors. [電信特考]**

(Sol.)  $V(b) = 0$ ,  $V(a) = V_0 \Rightarrow C_1 \ln(b) + C_2 = 0$ ,  $C_1 \ln(a) + C_2 = V_0$

$$C_1 = \frac{V_0}{\ln(b/a)}, \quad C_2 = -\frac{V_0 \ln b}{\ln(b/a)}, \quad \therefore V(r) = \frac{V_0}{\ln(b/a)} \ln\left(\frac{r}{b}\right)$$



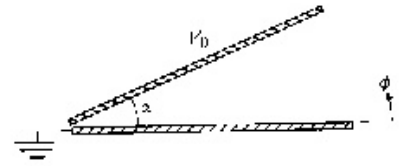
**Eg. Two infinite insulated conducting planes maintained at potentials 0 and  $V_0$  form a wedge-shaped configuration. Determine the potential distributions for the regions:**

**regions:** (a)  $0 < \phi < \alpha$   
 (b)  $\alpha < \phi < 2\pi$  . [中央光電所、成大電研]

(Sol.)

(a)  $V(\phi) = A\phi + B,$

$$\begin{cases} V(0) = 0 \Rightarrow B_0 = 0 \\ V(\alpha) = V_0 = A_0\alpha \Rightarrow A_0 = \frac{V_0}{\alpha} \Rightarrow V(\phi) = \frac{V_0}{\alpha}\phi, 0 \leq \phi \leq \alpha \end{cases}$$



(b)  $\begin{cases} V(\alpha) = V_0 = A_1\alpha + B_1 \\ V(2\pi) = 0 = 2\pi A_1 + B_1 \end{cases} \Rightarrow A_1 = -\frac{V_0}{2\pi - \alpha}, B_1 = \frac{2\pi V_0}{2\pi - \alpha} \Rightarrow V(\phi) = \frac{V_0}{2\pi - \alpha}(2\pi - \phi),$   
 $\alpha \leq \phi \leq 2\pi$

**Eg. An infinitely long, thin, conducting circular tube of radius  $b$  is split in two halves. The upper half is kept at a potential  $V=V_0$  and the lower half at  $V=-V_0$ . Determine the potential distributions both inside and outside the tube.**

(Sol.)  $V(b, \phi) = \begin{cases} V_0, 0 < \phi < \pi \\ -V_0, \pi < \phi < 2\pi \end{cases}$

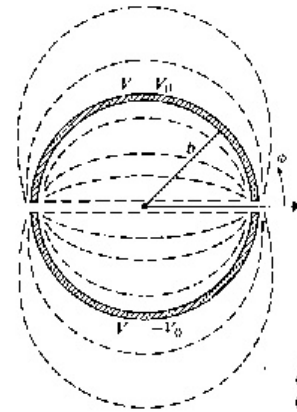
(a) Inside the tube:

$$r < b \Rightarrow V_n(r, \phi) = A_n r^n \sin n\phi \Rightarrow V(r, \phi) = \sum_{n=1}^{\infty} A_n r^n \sin n\phi$$

$$r = b \Rightarrow \sum_{n=1}^{\infty} A_n b^n \sin n\phi = \begin{cases} V_0, 0 < \phi < \pi \\ -V_0, \pi < \phi < 2\pi \end{cases}$$

$$\Rightarrow A_n = \begin{cases} \frac{4V_0}{n\pi b^n}, n : \text{odd} \\ 0, n : \text{even} \end{cases}$$

$$\Rightarrow V(r, \phi) = \frac{4V_0}{\pi} \sum_{n=\text{odd}} \frac{1}{n} \left(\frac{r}{b}\right)^n \sin n\phi, r < b$$



(b) Outside the tube:  $r > b \Rightarrow V_n(r, \phi) = B_n r^{-n} \sin n\phi \Rightarrow V(r, \phi) = \sum_{n=1}^{\infty} B_n r^{-n} \sin n\phi$

$$r = b \Rightarrow \sum_{n=1}^{\infty} B_n b^{-n} \sin n\phi = \begin{cases} V_0, 0 < \phi < \pi \\ -V_0, \pi < \phi < 2\pi \end{cases}$$

$$\Rightarrow B_n = \begin{cases} \frac{4V_0 b^n}{n\pi}, n : \text{odd} \\ 0, n : \text{even} \end{cases}$$

$$\Rightarrow V(r, \phi) = \frac{4V_0}{\pi} \sum_{n=\text{odd}} \frac{1}{n} \left(\frac{b}{r}\right)^n \sin n\phi, r > b$$

**Eg. A long, grounded conducting cylinder of radius  $b$  is placed along the  $z$ -axis in an initially uniform electric field  $\vec{E} = \hat{x}E_0$ . Determine potential distribution  $V(r, \phi)$  and electric field intensity  $\vec{E}(r, \phi)$  outside the cylinder. Show that the electric field intensity at the surface of the cylinder may be twice as high as that in the distance, which may cause a local breakdown or corona (St. Elmo's fire.)**  
[中央光電所、高考]

$$\text{(Sol.) } V(r, \phi) = -E_0 r \cos \phi + \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi \quad (\text{At } r \gg b, \vec{E} = \hat{x}E_0, V = -E_0 r \cos \phi)$$

$$\text{At } r = b : V(r, \phi) = -E_0 b \cos \phi + \sum_{n=1}^{\infty} B_n b^{-n} \cos n\phi = 0 \Rightarrow B_1 = E_0 b^2, B_n = 0 \text{ for } n \neq 1.$$

$$\text{Outside the cylinder, } r \geq b : V(r, \phi) = -E_0 r \left(1 - \frac{b^2}{r^2}\right) \cos \phi$$

$$\vec{E}(r, \phi) = -\nabla V = \hat{a}_r E_0 \left(\frac{b^2}{r^2} + 1\right) \cos \phi + \hat{a}_\phi E_0 \left(\frac{b^2}{r^2} - 1\right) \sin \phi$$

**Eg. A long dielectric cylinder of radius  $b$  and dielectric constant  $\epsilon_r$ , is placed along the  $x$ -axis in an initially uniform electric field  $\vec{E} = \hat{x}E_0$ . Determine  $V(r, \phi)$  and  $\vec{E}(r, \phi)$  both inside and outside the dielectric cylinder.**

$$\text{(Sol.) For } r \geq b, V_0(r, \phi) = -E_0 r \cos \phi + \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi$$

$$r \leq b, V_i(r, \phi) = \sum_{n=1}^{\infty} A_n r^n \cos n\phi$$

$$\begin{aligned} V_0(b, \phi) = V_i(b, \phi) & \Rightarrow -E_0 b + B_1 b^{-1} = A_1 b, \quad B_n b^{-n} = A_n b^n, \quad n \neq 1 \\ -\frac{\partial V_0}{\partial r}|_{r=b} = -\epsilon_r \frac{\partial V_i}{\partial r}|_{r=b} & \Rightarrow E_0 + B_1 b^{-2} = -\epsilon_r A_1, \quad n B_n b^{-(n+1)} = -\epsilon_r n A_n b^{n-1}, \quad n \neq 1 \end{aligned}$$

$$\Rightarrow \begin{cases} A_1 = \frac{-2E_0}{\epsilon_r + 1}, & B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 1} b^2 E_0 \\ A_n = B_n = 0 \text{ for } n \neq 1 \end{cases} \Rightarrow \begin{cases} V_0(r, \phi) = -\left(1 - \frac{\epsilon_r - 1}{\epsilon_r + 1} \cdot \frac{b^2}{r^2}\right) E_0 r \cos \phi \\ V_i(r, \phi) = -\frac{2}{\epsilon_r + 1} E_0 r \cos \phi \end{cases}$$

$$\vec{E} = -\nabla V = -\hat{a}_r \frac{\partial V}{\partial r} - \hat{a}_\phi \frac{\partial V}{r \partial \phi} \Rightarrow \begin{cases} \vec{E}_0 = \hat{a}_r E_0 \left(1 + \frac{\epsilon_r - 1}{\epsilon_r + 1} \cdot \frac{b^2}{r^2}\right) \cos \phi - \hat{a}_\phi E_0 \left(1 - \frac{\epsilon_r - 1}{\epsilon_r + 1} \cdot \frac{b^2}{r^2}\right) \sin \phi \\ \vec{E}_i = \frac{2}{\epsilon_r + 1} E_0 (\hat{a}_r \cos \phi - \hat{a}_\phi \sin \phi) \end{cases}$$

#### 4-4 Boundary-Value Problems in Spherical Coordinates

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Assume  $\phi$ -independent:  $\frac{\partial^2}{\partial \phi^2} = 0$ ,  $V(r, \theta) = R(r)\Theta(\theta)$

$$R^2 \frac{d^2 \Gamma(R)}{dR^2} + 2R \frac{d\Gamma(R)}{dR} - k^2 \Gamma(R) = 0, \quad \frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta(\theta)}{d\theta} \right] + n(n+1)\Theta(\theta)\sin(\theta) = 0, \text{ and}$$

$$k^2 = n(n+1) \rightarrow \Gamma(r) = A_n r^n + B_n r^{-n-1}, \quad \Theta(\theta) = P_n(\cos \theta) \rightarrow V(r, \theta) = [A_n r^n + B_n r^{-n-1}] P_n(\cos \theta)$$

Table of Legendre's Polynomials

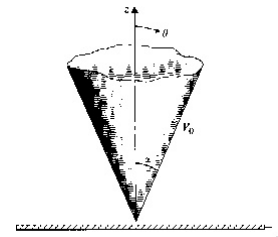
$n$	$P_n(\cos \theta)$
0	1
1	$\cos \theta$
2	$\frac{1}{2}(3 \cos^2 \theta - 1)$
3	$\frac{1}{2}(5 \cos^2 \theta - 3 \cos \theta)$

**Ex. An infinite conducting cone of half-angle  $\alpha$  is maintained at potential  $V_0$  and insulated from a grounded conducting plane. Determine (a) the potential distribution  $V(\theta)$  in the region  $\alpha < \theta < \pi/2$ , (b) the electric field intensity in the region  $\alpha < \theta < \pi/2$ , (c) the charge densities on the cone surface and on the grounded plane.**

(Sol.)

$$\frac{d}{d\theta} \left( \sin \theta \frac{dV}{d\theta} \right) = 0, \quad \frac{dV}{d\theta} = \frac{C_1}{\sin \theta} \Rightarrow V(\theta) = C_1 \ell n \left( \tan \frac{\theta}{2} \right) + C_2$$

$$\begin{aligned} \text{(a)} \quad V(\alpha) &= C_1 \ell n \left( \tan \frac{\alpha}{2} \right) + C_2 = V_0 \\ V\left(\frac{\pi}{2}\right) &= C_1 \ell n \left( \tan \frac{\pi}{4} \right) + C_2 = 0 \end{aligned} \Rightarrow \begin{aligned} C_1 &= \frac{V_0}{\ell n \left[ \tan \left( \frac{\alpha}{2} \right) \right]} \\ C_2 &= 0 \end{aligned} \Rightarrow V(\theta) = \frac{V_0 \ell n \left[ \tan \left( \frac{\theta}{2} \right) \right]}{\ell n \left[ \tan \left( \frac{\alpha}{2} \right) \right]}$$



$$\begin{aligned} \text{(b)} \quad \vec{E} &= -\hat{a}_\theta \frac{dV}{R d\theta} = -\hat{a}_\theta \frac{V_0}{R \ell n \left[ \tan \left( \frac{\alpha}{2} \right) \right] \sin \theta}, \quad \text{(c)} \\ \theta = \alpha : \rho_s &= \epsilon_0 E(\alpha) = \frac{\epsilon_0 V_0}{R \ell n \left[ \tan \left( \frac{\alpha}{2} \right) \right] \sin \theta} \\ \theta = \frac{\pi}{2} : \rho_s &= -\epsilon_0 E\left(\frac{\pi}{2}\right) = -\frac{\epsilon_0 V_0}{R \ell n \left[ \tan \left( \frac{\alpha}{2} \right) \right]} \end{aligned}$$



**Eg. An uncharged conducting sphere of radius  $b$  is placed in an initially uniform electric field  $\vec{E} = \hat{z}E_0$ . Determine the potential distribution  $V(R,\theta)$  and the electric field intensity  $\vec{E}(R,\theta)$  after the introduction of the sphere. [中山電研]**

(Sol.)  $V(b,\theta) = 0$

If  $R \gg b, V(R,\theta) = -E_0 z = -E_0 R \cos \theta$

$$V(R,\theta) = \sum_{n=0}^{\infty} [A_n R^n + B_n R^{-(n+1)}] P_n(\cos \theta), \quad R \geq b$$

$$\left( \begin{array}{l} A_n = 0, n \neq 1 \\ A_1 = -E_0 \end{array} \right)$$

↓

$$= -E_0 R P_1(\cos \theta) + \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos \theta), \quad R \geq b$$

(sphere is uncharged,  $B_0 = 0$ )

↓

$$= \left( \frac{B_1}{R^2} - E_0 R \right) \cos \theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos \theta), \quad R \geq b$$

$$R = b, \quad 0 = \left( \frac{B_1}{b^2} - E_0 b \right) \cos \theta + \sum_{n=2}^{\infty} B_n b^{-(n+1)} P_n(\cos \theta) \Rightarrow B_1 = E_0 b^3, \quad B_n = 0, \quad n \geq 2,$$

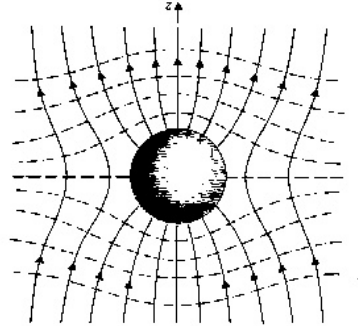
$$\therefore V(R,\theta) = -E_0 \left[ 1 - \left( \frac{b}{R} \right)^3 \right] R \cos \theta, \quad R \geq b$$

$$\vec{E}(R,\theta) = \hat{a}_R E_R + \hat{a}_\theta E_\theta = -\nabla V(R,\theta) = \hat{a}_R \left( -\frac{\partial V}{\partial R} \right) + \hat{a}_\theta \left( -\frac{\partial V}{R \partial \theta} \right)$$

$$= \hat{a}_R E_0 \left[ 1 + 2 \left( \frac{b}{R} \right)^3 \right] \cos \theta - \hat{a}_\theta E_0 \left[ 1 - \left( \frac{b}{R} \right)^3 \right] \sin \theta, \quad R \geq b$$

A dipole moment  $\vec{P} = \hat{z} 4\pi \epsilon_0 b^2 E_0$  is at the center of the sphere. Surface charge

density is  $\rho_s(\theta) = \epsilon_0 E_R|_{R=b} = 3\epsilon_0 E_0 \cos \theta$



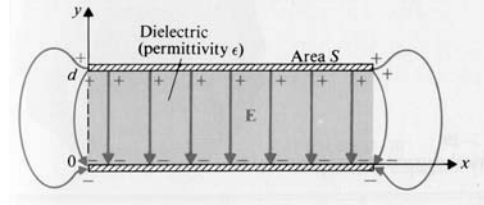
## 4-5 Capacitors and Capacitances

$$Q=CV \Leftrightarrow C=Q/V$$

**Eg. A parallel-plane capacitor consists of two parallel conducting plates of area  $S$  separate by uniform distance  $d$ , the space between the plates is filled with a dielectric of a constant permittivity. Determine the capacitance.**

$$\text{(Sol.) } \rho_s = \frac{Q}{S}, \quad \vec{E} = -\hat{y} \frac{\rho_s}{\epsilon} = -\hat{y} \frac{Q}{\epsilon S}$$

$$V = -\int_{y=0}^{y=d} \vec{E} \cdot d\vec{l} = \int_0^d \left( -\hat{y} \frac{Q}{\epsilon S} \right) \cdot (\hat{y} dy) = \frac{Q}{\epsilon S} d$$



$$C = \frac{Q}{V} = \epsilon \frac{S}{d}. \text{ In this problem, } \vec{E} = -\hat{y} \frac{V}{d}$$

$\Rightarrow$  Surface charge densities on the upper and conducting planes are  $\rho_s$  and  $-\rho_s$ ,  
 $\rho_s = \epsilon E_y = \epsilon V/d$ .

**Eg. The space between a parallel-plate capacitor of area  $S$  is filled with dielectric whose permittivity varies linearly from  $\epsilon_1$  at one plate ( $y=0$ ) to  $\epsilon_2$  at the other plate ( $y=d$ ). Neglecting all the edge effect, find the capacitance. [台科大電子所]**

$$\text{(Sol.) Assume } Q \text{ on plate at } y=d, \quad \epsilon = \frac{\epsilon_2 - \epsilon_1}{d} y + \epsilon_1$$

$$\vec{E} = -\hat{y} \frac{\rho_s}{\epsilon}, \quad \rho_s = \frac{Q}{S} \Rightarrow V = -\int_0^d \vec{E} \cdot d\vec{l} = \frac{Qd \ln(\epsilon_2/\epsilon_1)}{S(\epsilon_2 - \epsilon_1)} \Rightarrow C = \frac{Q}{V} = \frac{S(\epsilon_2 - \epsilon_1)}{d \ln(\frac{\epsilon_2}{\epsilon_1})}$$

**Eg. A cylindrical capacitor consists of an inner conductor of radius  $a$  and an outer conductor of radius  $b$  is filled with a dielectric of permittivity  $\epsilon$ , and the length of the capacitor is  $L$ . Determine the capacitance of this capacitor.**

$$\text{(Sol.) } \vec{E} = \hat{a}_r E_r = \frac{Q}{2\pi\epsilon L r}$$

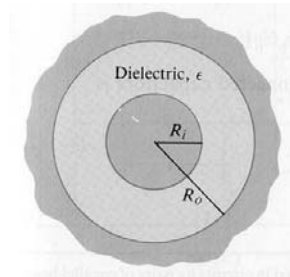
$$V_{ab} = -\int_{r=b}^{r=a} \vec{E} \cdot d\vec{l} = -\int_b^a \left( \hat{a}_r \frac{Q}{2\pi\epsilon L r} \right) \cdot (\hat{a}_r dr) = \frac{Q}{2\pi\epsilon L} \ln\left(\frac{b}{a}\right), \quad C = \frac{Q}{V_{ab}} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$$

**Ex. A spherical capacitor consists of an inner conducting sphere of radius  $R_i$  and an outer conductor with a spherical wall of radius  $R_o$ . The space in between them is filled with dielectric of permittivity  $\epsilon$ . Determine the capacitance.**

(Sol.)  $\vec{E} = \hat{a}_r E_r = \hat{a}_r \frac{Q}{4\pi\epsilon R^2}$

$$V = -\int_{R_o}^{R_i} \vec{E} \cdot \hat{a}_r dR = -\int_{R_o}^{R_i} \frac{Q}{4\pi\epsilon R^2} dR = \frac{Q}{4\pi\epsilon} \left( \frac{1}{R_i} - \frac{1}{R_o} \right)$$

$$C = \frac{Q}{V} = \frac{4\pi\epsilon}{\frac{1}{R_i} - \frac{1}{R_o}}$$



For an isolating conductor sphere of a radius:  $R_i, R_o \rightarrow \infty, C = 4\pi\epsilon R_i$

**Ex. Assuming the earth to be a large conducting sphere (radius= $6.37 \times 10^3 km$ ) surrounded by air, find (a) the capacitance of the earth, (b) the maximum charge that can exist on the earth before the air breaks down.**

(Sol.) (a)  $C = 4\pi\epsilon_0 R = 4\pi \times \frac{1}{36\pi} \times 10^{-9} \times (6.37 \times 10^3 \times 10^3) = 7.08 \times 10^{-4} \text{ (F)}$

(b)  $E_b = 3 \times 10^6 = \frac{Q_{Max}}{4\pi\epsilon_0 R^2} \Rightarrow Q_{Max} = 1.35 \times 10^{10} \text{ (C)}$

**Ex. Determine the capacitance of an isolated conducting sphere of radius  $b$  that is coated with a dielectric layer of uniform thickness  $d$ , the dielectric has an electric susceptibility  $\chi_e$ .**

(Sol.)  $b < R < b + d, \vec{E} = \hat{a}_r \frac{Q}{4\pi\epsilon_0(1 + \chi_e)R^2}, b + d < R, \vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}$

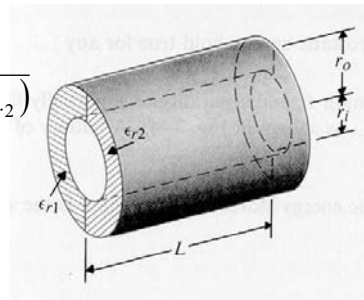
$$V = -\int_{\infty}^b \vec{E} \cdot d\vec{l} = \frac{Q}{4\pi\epsilon_0(1 + \chi_e)} \left( \frac{\chi_e}{b + d} + \frac{1}{b} \right), C = \frac{Q}{V} = \frac{4\pi\epsilon_0(1 + \chi_e)}{\frac{\chi_e}{b + d} + \frac{1}{b}}$$

**Ex. A cylindrical capacitor of length  $L$  consists of coaxial conducting surface of radii  $r_i$  and  $r_o$ . Two dielectric media of different dielectric constants  $\epsilon_{r1}$  and  $\epsilon_{r2}$ , and fill the space between the conducting surface. Determine the capacitance. [台大物理所、高考電機技師]**

(Sol.)  $\pi r L (\epsilon_0 \epsilon_{r1} + \epsilon_0 \epsilon_{r2}) E = \rho_l L \Rightarrow E = \frac{\rho_l}{\pi r \epsilon_0 (\epsilon_{r1} + \epsilon_{r2})}$

$$V = -\int_{r_o}^{r_i} E dr = \frac{\rho_l}{\pi \epsilon_0 (\epsilon_{r1} + \epsilon_{r2})} \ln \left( \frac{r_o}{r_i} \right)$$

$$C = \frac{\rho_l L}{V} = \frac{\pi \epsilon_0 (\epsilon_{r1} + \epsilon_{r2}) L}{\ln(r_o/r_i)}$$



**Eg. The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are  $r_i$  and  $r_o$  respectively. The space between two conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are  $\epsilon_{r1}$  for  $r_i < r < b$  and  $\epsilon_{r2}$  for  $b < r < r_o$ . Determine its capacitance per unit length.**

$$\text{(Sol.) } \vec{E}_1 = \hat{a}_r \frac{\rho_l}{2\pi\epsilon_0\epsilon_{r1}r}, \quad r_i < r < b, \quad \vec{E}_2 = \hat{a}_r \frac{\rho_l}{2\pi\epsilon_0\epsilon_{r2}r}, \quad b < r < r_o$$

$$V = -\int_{r_o}^{r_i} \vec{E} \cdot d\vec{r} = \frac{\rho_l}{2\pi\epsilon_0} \left[ \frac{1}{\epsilon_{r1}} \ln\left(\frac{b}{r_i}\right) + \frac{1}{\epsilon_{r2}} \ln\left(\frac{r_o}{b}\right) \right],$$

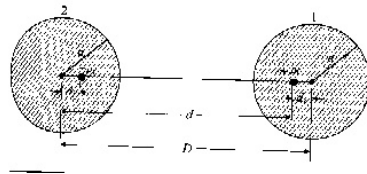
$$C = \frac{\rho_l}{V} = \frac{2\pi\epsilon_0}{\frac{1}{\epsilon_{r1}} \ln\left(\frac{b}{r_i}\right) + \frac{1}{\epsilon_{r2}} \ln\left(\frac{r_o}{b}\right)} \quad (F/m)$$

**Eg. Determine the capacitance per unit length between two long, parallel, circular conducting wires of radius  $a$ . The axes of the wires are separated by a distance  $D$ . [台大電研]**

$$\text{(Sol.) } V_2 = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{a}{d}, \quad V_1 = -\frac{\rho_l}{2\pi\epsilon_0} \ln \frac{a}{d}$$

$$C = \frac{\rho_l}{V_1 - V_2} = \frac{\pi\epsilon_0}{\ln(d/a)}, \quad d = D - d_i = D - \frac{a^2}{d}, \quad d = \frac{1}{2} \left( D + \sqrt{D^2 - 4a^2} \right)$$

$$C = \frac{\pi\epsilon}{\ln \left[ (D/2a) + \sqrt{(D/2a)^2 - 1} \right]} = \frac{\pi\epsilon}{\cosh^{-1}(D/2a)} \quad (F/m)$$



**Eg. A straight conducting wire of radius  $a$  is parallel to and at height  $h$  from the surface of the earth. Assume that the earth is perfectly conducting, determine the capacitance and the force per unit length between the wire and the earth.**

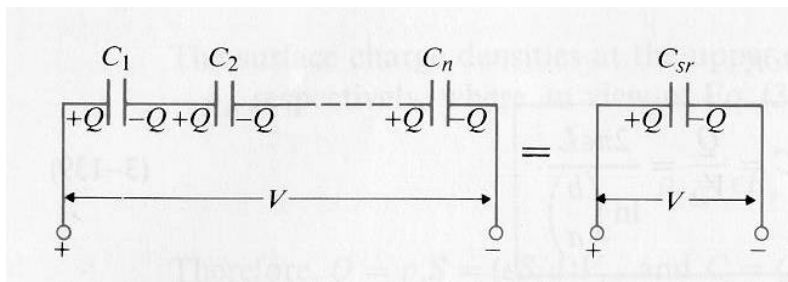
$$\text{(Sol.) } D=2h, \quad C = \frac{\pi\epsilon_0}{\cosh^{-1}(D/2a)} = \frac{\pi\epsilon_0}{\cosh^{-1}(h/a)} \quad (F/m)$$

**Eg. A capacitor consists of two coaxial metallic cylindrical surface a length 30mm and radii 5mm and 7mm. The dielectric between the surfaces has a relative permittivity  $\epsilon_r=2+(4/r)$ , where  $r$  is measured in mm. Determine the capacitance of the capacitor.**

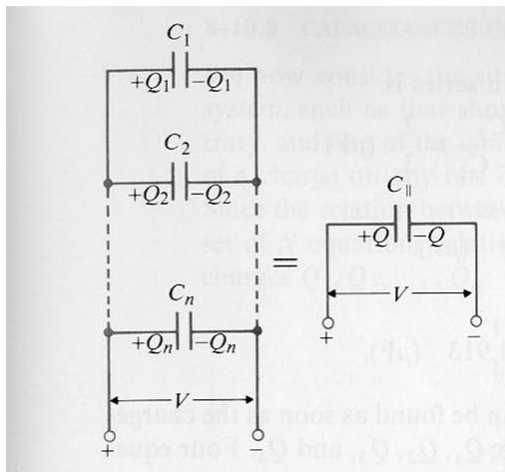
(Sol.)  $\vec{E} = \hat{a}_r \frac{\rho_l}{2\pi\epsilon r} = \hat{a}_r \frac{\rho_l}{2\pi\epsilon_0 \left(2 + \frac{4}{r}\right) r} = \hat{a}_r \frac{\rho_l}{4\pi\epsilon_0 (r+2)}$

$$V = -\int_{r_0}^{r_i} \vec{E} \cdot d\vec{r} = \frac{\rho_l}{4\pi\epsilon_0} \ln(r+2) \Big|_{r=5}^{r=7} = \frac{\rho_l}{4\pi\epsilon_0} \ln\left(\frac{9}{7}\right), \quad C = \frac{\rho_l L}{V} = \frac{4\pi\epsilon_0 L}{\ln\left(\frac{9}{7}\right)} \approx 1500\epsilon_0$$

**Series or parallel connection of capacitance:**



$$V = \frac{Q}{C_{sr}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \dots + \frac{Q}{C_n} \Rightarrow \frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n}$$



$$\begin{aligned} Q &= Q_1 + Q_2 + \dots + Q_n \\ \Rightarrow C_{||} V &= C_1 V + C_2 V + \dots + C_n V \\ \Rightarrow C_{||} &= C_1 + C_2 + \dots + C_n \end{aligned}$$

#### 4-6 Electrostatic Energy

To remove  $Q_1$  from infinite to a distance  $R_{12}$  from  $Q_2$ , the amount of work required is

$$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}} = Q_1 \frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1 V_1 = \frac{1}{2}(Q_1 V_1 + Q_2 V_2)$$

$$\xrightarrow{\text{induction method}} W_e = \frac{1}{2} \sum_{k=1}^N Q_k V_k, \quad \text{where } V_k = \frac{1}{4\pi\epsilon_0} \sum_{j=1(j \neq k)}^N \frac{Q_j}{R_{jk}}$$

**Eg. Determine the work done in carrying a  $-2\mu\text{C}$  charge from  $P_1(2,1,-1)$  to  $P_2(8,2,-1)$  in the field  $\vec{E} = \hat{x}y + \hat{y}x$  along (a) the parabola  $x=2y^2$ , (b) the straight line joining  $P_1$  and  $P_2$ .**

(Sol.)  $d\vec{l} = \hat{x}dx + \hat{y}dy$ ,  $W = -q \int \vec{E} \cdot d\vec{l} = -q \cdot (\int ydx + xdy)$

(a) Along  $x = 2y^2$ ,  $dx = 4ydy \Rightarrow W = -q \int_1^2 6y^2 dy = -14q = 28 (\mu\text{J})$ .

(b) Along  $x = 6y - 4$ ,  $dx = 6dy \Rightarrow W = -q \int_1^2 (12y - 4) dy = -14q = 28 (\mu\text{J})$

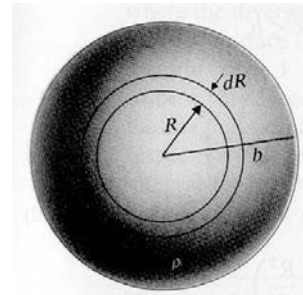
**Eg. Find the energy required to assemble a uniform charge of radius  $b$  and volume charge density  $\rho$ . [清大電研]**

(Sol.)  $V_R = \frac{Q_R}{4\pi\epsilon_0 R}$      $Q_R = \rho \frac{4}{3} \pi R^3$

$$dQ_R = \rho 4\pi R^2 dR, \quad dW = V_R dQ_R = \frac{4\pi}{3\epsilon_0} \rho^2 R^4 dR$$

$$W = \int dW = \frac{4\pi}{3\epsilon_0} \rho^2 \int_0^b R^4 dR = \frac{4\pi\rho^2 b^5}{15\epsilon_0}$$

$$Q = \rho \frac{4\pi}{3} b^3, \quad W = \frac{3Q^2}{20\pi\epsilon_0 b} \quad (J)$$



**Eg. Show that the stored electric energy is**  $W_e = \frac{1}{2} \iiint_{v'} \rho V dv = \frac{1}{2} \iiint_{v'} (\nabla \cdot \vec{D}) V dv$

(Proof)  $\therefore \nabla \cdot (V \vec{D}) = V \nabla \cdot \vec{D} + \vec{D} \cdot \nabla V$

$\therefore W_e = \frac{1}{2} \iiint_{v'} \nabla \cdot (V \vec{D}) dv - \frac{1}{2} \iiint_{v'} \vec{D} \cdot \nabla V dv = \frac{1}{2} \oint_{s'} V \vec{D} \cdot \hat{a}_n ds + \frac{1}{2} \iiint_{v'} \vec{D} \cdot \vec{E} dv$

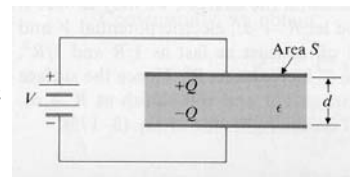
When  $R \rightarrow \infty, S \propto R^2, V \propto \frac{1}{R}, |\vec{D}| \propto \frac{1}{R^2} \Rightarrow \frac{1}{2} \oint_{s'} V \vec{D} \cdot \hat{a}_n ds \rightarrow 0 \Rightarrow W_e = \frac{1}{2} \iiint_{v'} \vec{D} \cdot \vec{E} dv$

If  $\vec{D} = \epsilon \vec{E}$ , then  $W_e = \frac{1}{2} \iiint_{v'} \epsilon |\vec{E}|^2 dv = \frac{1}{2} \iiint_{v'} \frac{|\vec{D}|^2}{\epsilon} dv = \iiint_{v'} w_e dv$

- Note:** 1. SI unit for energy: Joule(J) and  $1 eV = 1.6 \times 10^{-19} J$ .
- 2. Work (or energy) is a scalar, not a vector.

**Electrostatic energy density:**  $w_e = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1}{2} \epsilon |\vec{E}|^2 = \frac{|\vec{D}|^2}{2\epsilon}$

**Eg. A parallel-plate capacitor of area  $S$  and separation  $d$  is charge by a  $d-c$  voltage source  $V$ . The permittivity of the dielectric is  $\epsilon$ . Find the stored electrostatic energy.**



(Sol.)  $E = \frac{V}{d}, W_e = \frac{1}{2} \iiint_{v'} \epsilon \left(\frac{V}{d}\right)^2 dv = \frac{1}{2} \epsilon \left(\frac{V}{d}\right)^2 (Sd) = \frac{1}{2} \left(\epsilon \frac{S}{d}\right) V^2$

**Eg. Use energy formulas to find the capacitance of a cylindrical capacitance having a length  $L$ , an inner conductor of radius  $a$ , an outer conductor of inner radius  $b$ , and dielectric of permittivity  $\epsilon$ .**

(Sol.)  $\vec{E} = \hat{a}_r \frac{Q}{2\pi\epsilon Lr}, W_e = \frac{1}{2} \int_a^b \epsilon \left(\frac{Q}{2\pi\epsilon Lr}\right)^2 (L2\pi r dr) = \frac{Q^2}{4\pi\epsilon L} \int_a^b \frac{dr}{r} = \frac{Q^2}{4\pi\epsilon L} \ln\left(\frac{b}{a}\right),$

$\frac{Q^2}{2C} = \frac{Q^2}{4\pi\epsilon L} \ln\left(\frac{b}{a}\right) \Rightarrow C = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$

**Eg. Find the electrostatic energy stored in the region of space  $R > b$  around an electric dipole of moment  $p$ .**

(Sol.)  $\vec{E} = \frac{\vec{P}}{4\pi\epsilon_0 R^2} (\hat{a}_R 2 \cos \theta + \hat{a}_\theta \sin \theta), W = \frac{1}{2} \epsilon_0 \iiint_v |E|^2 dv' = \frac{p^2}{12\pi\epsilon_0 b^3}$

#### 4-7 Electrostatic Forces and Torques

##### Electrostatic force and torque due to the fixed charge:

$dW = \vec{F}_Q \cdot d\vec{l}$  is mechanic work done by the system, it costs the stored energy.

$$\therefore dW_e = -dW = -\vec{F}_Q \cdot d\vec{l} = (\nabla W_e) \cdot d\vec{l}, \therefore \vec{F}_Q = -\nabla W_e \quad (N)$$

$$\left( \vec{F}_Q \right)_l = -\frac{\partial W_e}{\partial l} = -\frac{\partial}{\partial l} \left( \frac{Q^2}{2C} \right) = \frac{Q^2}{2C^2} \frac{\partial C}{\partial l}, \quad dW = (T_Q)_z d\phi \Rightarrow (T_Q)_z = -\frac{\partial W_e}{\partial \phi}$$

##### Electrostatic force and torque due to the fixed potential:

$$dW_s = \sum_k V_k dQ_k, \quad dW = \vec{F}_v \cdot d\vec{l}, \quad dW_e = \frac{1}{2} \sum_k V_k dQ_k = \frac{1}{2} dW_s$$

$$dW + dW_e = dW_s \Rightarrow dW = \frac{1}{2} dW_s = dW_e = \vec{F}_v \cdot d\vec{l} = (\nabla W_e) \cdot d\vec{l}$$

$$\therefore \vec{F}_v = \nabla W_e, \quad (T_v)_z = \frac{\partial W_e}{\partial \phi}, \quad \left( \vec{F}_v \right)_l = \frac{\partial W_e}{\partial l} = \frac{\partial}{\partial l} \left( \frac{1}{2} CV^2 \right) = \frac{V^2}{2} \frac{\partial C}{\partial l} = \frac{Q^2}{2C^2} \frac{\partial C}{\partial l}$$

**Eg. Determine the force on the conducting plates of a charged parallel-plate capacitor. The plates have an area  $S$  and separate in air by a distance  $x$ .**

(Sol.) (a) Assuming fixed charge,  $W_e = \frac{1}{2} QV = \frac{1}{2} QE_x x$ ,

$$(F_Q)_x = -\frac{\partial}{\partial x} \left( \frac{1}{2} QE_x x \right) = -\frac{Q^2}{2\epsilon_0 S}$$

(b) Assuming the fixed potential,

$$(F_v)_x = \frac{\partial W_e}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{2} CV^2 \right) = \frac{V^2}{2} \frac{\partial}{\partial x} \left( \frac{\epsilon_0 S}{x} \right) = -\frac{\epsilon_0 S V^2}{2x^2}$$

$$\therefore Q = CV = \frac{\epsilon_0 S V}{x}, \therefore (F_Q)_x = (F_v)_x$$

**Eg. A parallel-plate capacitor of width  $w$ , length  $L$ , and separation  $d$  is partially filled with a dielectric medium of dielectric constants  $\epsilon_r$ . A battery of  $V_0$  volts is connected between the plates. (a) Find  $\vec{D}$ ,  $\vec{E}$ ,  $\rho_s$  in each region. (b) Find distance  $x$  such that the electrostatic energy stored in each region is the same. [台大電研]**

$$(Sol.) (a) \quad \vec{E}_1 = -\hat{y} \frac{V_0}{d}, \vec{D}_1 = -\hat{y} \epsilon_0 \epsilon_r \frac{V_0}{d}, \rho_{s1} = \epsilon_0 \epsilon_r \frac{V_0}{d}$$



$$\vec{E}_2 = -\hat{y} \frac{V_0}{d}, \vec{D}_2 = -\hat{y} \epsilon_0 \frac{V_0}{d}, \rho_{s2} = \epsilon_0 \frac{V_0}{d} \quad (b) \quad \frac{W_{e1}}{W_{e2}} = \frac{\epsilon_r x}{L-x} = 1 \Rightarrow x = \frac{L}{\epsilon_r + 1}$$



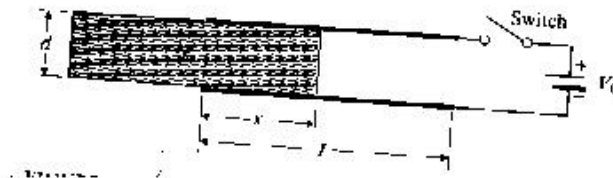
**Eg. A parallel-plate capacitor of width  $w$ , length  $L$ , and separation  $d$  has a solid dielectric slab of permittivity  $\epsilon$  in the space between the plates. The capacitor is charged to a voltage  $V_0$  by a battery. Assuming that the dielectric slab is withdrawn to the position shown, determine the force action on the slab. (a) with the switch closed, (b) after the switch is first opened. [台大電研、清大電研]**

(Sol.) (a)

$$W_e = \frac{1}{2} C V_0^2, \quad C = \frac{w}{d} [\epsilon x + \epsilon_0 (L - x)] \Rightarrow \bar{F}_x = \nabla W_e = \hat{x} \frac{V_0^2}{2} \frac{\partial C}{\partial x} = \hat{x} \frac{V_0^2 w}{2d} (\epsilon - \epsilon_0)$$

(b)

$$W_s = \frac{Q^2}{2C}, \quad \bar{F}_x = -\nabla W_s = -\hat{x} \frac{Q^2}{2} \frac{\partial}{\partial x} \left( \frac{1}{C} \right) = \hat{x} \frac{V_0^2 w}{2d} (\epsilon - \epsilon_0)$$



**Eg. The conductors of an isolated two-wire transmission line, each of radius  $b$ , are spaced at a distance  $D$  apart. Assuming  $D \gg b$  and a voltage  $V_0$  between the lines, find the force per unit length on the lines.**

(Sol.)

$$\bar{E} = \hat{x} \left[ \frac{\rho_l}{2\pi\epsilon_0 x} + \frac{\rho_l}{2\pi\epsilon_0 (D-x)} \right], \quad V_0 = V_1 - V_2 = \int_b^{D-b} \bar{E} \cdot d\hat{x} = \frac{\rho_l}{\pi\epsilon_0} \ln \frac{D-b}{b} \approx x \frac{\rho_l}{\pi\epsilon_0} \ln \frac{D}{b}$$

$$C' = \frac{\rho_l}{V_0} = \frac{\pi\epsilon_0}{\ln(D/b)} \quad (F/m), \quad \bar{F}' = \nabla W_e = \hat{x} \frac{V_0^2}{2} \frac{\partial C'}{\partial D} = -\hat{x} \frac{\pi\epsilon_0 V_0^2}{2D [\ln(D/b)]^2}$$

#### 4-8 Resistors and Resistances

**Ohm's law:**  $V=RI$

$$V = E\ell \Rightarrow E = \frac{V_{12}}{\ell}, I = \iint_s \vec{J} \cdot d\vec{S} = JS \Rightarrow J = \frac{I}{S} = \sigma \frac{V}{\ell} \Rightarrow V = \left( \frac{\ell}{\sigma S} \right) I = RI$$

$$\therefore R = \frac{\ell}{\sigma S}, G = \frac{1}{R} = \sigma \frac{S}{\ell}$$

**Power dissipation:**  $P = \iiint_{v'} \vec{E} \cdot \vec{J} dv = \int \vec{E} \cdot d\vec{\ell} \iint_{ss} \vec{J} \cdot d\vec{S} = -VI = -I^2 R$

**Eg. A long round wire of radius  $a$  and conductivity  $\sigma$  is coated with a material of conductivity  $0.1\sigma$ . (a) What must be the thickness of the coating so that the resistance per unit length of the uncoated wire is reduced by 50%? (b) Assuming a total current  $I$  in the coated wire, find  $J$  and  $E$  in both the core and the coating material. [台科大電子所]**

(Sol.)  $R_1 = \frac{1}{\sigma\pi a^2}, R_2 = \frac{1}{\sigma\pi[(a+b)^2 - a^2]}$

(a)  $R_1 = R_2 \Rightarrow b = (\sqrt{11} - 1)a,$

(b)  $I_1 = I_2 = \frac{I}{2}, J_1 = \frac{I}{2\pi a^2} = \sigma E_1, J_2 = \frac{I}{2\pi[(a+b)^2 - b^2]} = 0.1\sigma E_2$

$\Rightarrow J_1 = 10J_2, E_1 = E_2$

**Eg. A  $d-c$  voltage of  $6V$  applied to the ends of  $1km$  of a conducting wire of  $0.5mm$  radius results in a current of  $1/6A$ . Find (a) the conductivity of the wire, (b) the electric field intensity of the wire, (c) the power dissipation in the wire, (d) the electron drift velocity, assuming electron mobility in the wire to be  $1.4 \times 10^{-3} (m^2 / V \cdot s)$ .**

(Sol.) (a)  $R = \frac{\ell}{\sigma S} = \frac{V}{I} \Rightarrow \sigma = \frac{\ell I}{SV} = 3.54 \times 10^7 (S/m),$  (b)  $E = \frac{V}{\ell} = 6 \times 10^{-3} (V/m),$  (c)

$P = VI = 1 \text{ Watt},$  (d)  $v_e = \mu E = 8.4 \times 10^{-6} (m/sec)$

**Calculation of resistance:**

$$\nabla^2 V = 0 \Rightarrow V \Rightarrow \vec{E} = -\nabla V \Rightarrow \vec{J} = \sigma \vec{E} \Rightarrow \vec{I} = \oiint \vec{J} ds \Rightarrow R = V / I$$

**Eg. A conducting material of uniform thickness  $h$  and conductivity  $\sigma$ , has the shape of a quarter of a flat circular washer, with inner radius  $a$  and outer radius  $b$ . Determine the resistance between the end faces. [清大電研]**

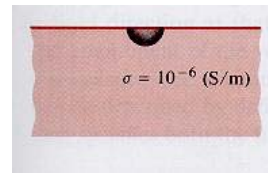
(Sol.)  $\nabla^2 V = 0, V=0$  at  $\phi = 0, V=V_0$  at  $\phi = \frac{\pi}{2}$

$$\frac{d^2 V}{d\phi^2} = 0, V=c_1\phi+c_2, V = \frac{2V_0}{\pi}\phi, \vec{J} = \sigma \vec{E} = -\sigma \nabla V = -\hat{a}_\phi \sigma \frac{\partial V}{r \partial \phi} = -\hat{a}_\phi \frac{2\sigma V_0}{\pi r}$$

$$I = \int_S \vec{J} \cdot d\vec{s} = \frac{2\sigma V_0}{\pi} h \int_a^b \frac{dr}{r} = \frac{2\sigma h V_0}{\pi} \ln \frac{b}{a}, R = \frac{V_0}{I} = \frac{\pi}{2\sigma h \ln\left(\frac{b}{a}\right)}$$

**Eg. A ground connection is made by burying a hemispherical conductor of radius  $25mm$  in the earth with its base up. Assuming the earth conductivity to  $\sigma=10^{-6} S/m$ , find the resistance of the conductor to far-away points in the ground. [交大電信所]**

(Sol.)  $\vec{J} = \hat{a}_R \frac{I}{2\pi R^2}, \vec{E} = \hat{a}_R \frac{I}{2\pi \sigma R^2} \Rightarrow V_0 = -\int_\infty^b E dR = \frac{I}{2\pi \sigma b}$

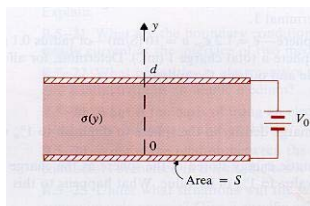


$$R = \frac{V_0}{I} = \frac{1}{2\pi \sigma b} = \frac{1}{2\pi(10^{-6})(25 \times 10^{-3})} = 6.36 \times 10^6.$$

**Eg. The space between two parallel conducting plates each having an area  $S$  is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from  $\sigma_1$  at one plate ( $y=0$ ) to  $\sigma_2$  at the other plate ( $y=d$ ). And  $d-c$  voltage  $V_0$  is applied across the plates. Determine the total resistance between the plates.**

(Sol.)  $\vec{J} = -\hat{y} J_0 \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = -\hat{y} \frac{J_0}{\sigma(y)}, \sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1) \frac{y}{d},$

$$V_0 = -\int_0^d \vec{E} \cdot \hat{y} dy = \frac{J_0 d}{\sigma_2 - \sigma_1} \ln \frac{\sigma_2}{\sigma_1}, R = \frac{V_0}{I} = \frac{V_0}{J_0 S} = \frac{d}{(\sigma_2 - \sigma_1) S} \ln \left( \frac{\sigma_2}{\sigma_1} \right)$$



**Relation between  $R$  and  $C$ :**  $C = \frac{Q}{V} = \frac{\oiint_S \vec{D} \cdot d\vec{s}}{-\int_L \vec{E} \cdot d\vec{l}} = \frac{\oiint_S \epsilon \vec{E} \cdot d\vec{s}}{-\int_L \vec{E} \cdot d\vec{l}},$

$$R = \frac{V}{I} = \frac{-\int_L \vec{E} \cdot d\vec{l}}{\oiint_S \vec{J} \cdot d\vec{s}} = \frac{-\int_L \vec{E} \cdot d\vec{l}}{\oiint_S \sigma \vec{E} \cdot d\vec{s}}, \therefore \boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}}$$

**Eg. Find the leakage resistance per unit length (a) between the inner and outer conductors of a coaxial cable that has an inner conductor of radius  $a$ , an outer conductor of inner radius  $b$ , and a medium with conductivity  $\sigma$ , and (b) of a parallel-wire transmission line consisting of wires of radius  $a$  separated by a distance  $D$  in a medium with conductivity  $\sigma$ . [台科大電研]**

(Sol.) (a)  $C = \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)}, R = \frac{\epsilon}{\sigma} \left(\frac{1}{C}\right) = \frac{1}{2\pi\sigma} \ln\left(\frac{b}{a}\right)$

(b)  $C = \frac{\pi\epsilon}{\cosh^{-1}\left(\frac{D}{2a}\right)}, R = \frac{\epsilon}{\sigma} \left(\frac{1}{C}\right) = \frac{1}{\pi\sigma} \cosh^{-1}\left(\frac{D}{2a}\right)$

**Eg. Find the resistance between two concentric spherical surfaces of radii  $R_1$  and  $R_2$  ( $R_1 < R_2$ ) if the space between the surfaces is filled with a homogeneous and isotropic material having a conductivity  $\sigma$ .**

(Sol.)  $C = \frac{4\pi\epsilon}{\frac{1}{R_1} - \frac{1}{R_2}}, RC = \frac{\epsilon}{\sigma} \Rightarrow R = \frac{1}{C} \cdot \frac{\epsilon}{\sigma} \Rightarrow R = \frac{1}{4\pi\sigma} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

#### 4-9 Inductors and Inductances

**Mutual flux:**  $\Phi_{12} = \iint_{S_2} \vec{B}_1 \cdot d\vec{S}_2 = L_{12}I_1$  (Wb)

**General mutual inductance:**  $L_{12} = \frac{N_2\Phi_{12}}{I_1} = \frac{\Lambda_{12}}{I_1}$  (H)

**Self-Inductance:**  $L_{11} = \frac{\Lambda_{11}}{I_1}$

**Eg. Assume that  $N$  turns of wire are tightly wound on a toroidal frame of a rectangular cross section. Then, assuming the permeability of the medium to be  $\mu_0$ , find the self-inductance of the toroidal coil. [台大電研]**

(Sol.)

$$d\vec{l} = \hat{a}_\phi r d\phi, \oint_C \vec{B} \cdot d\vec{l} = \int Br d\phi = 2\pi r B = \mu_0 NI \Rightarrow B = \frac{\mu_0 NI}{2\pi r}$$

$$\Rightarrow \Phi = \iint_S \vec{B} \cdot d\vec{s} = \frac{\mu_0 NI h}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 NI h}{2\pi} \ln\left(\frac{b}{a}\right), L = \frac{N\Phi}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

**Eg. Find the inductance per unit length of a very long solenoid with air core having  $n$  turns per unit length. And  $S$  is the cross-sectional area.**

(Sol.)  $B = \mu_0 nI, \Phi = BS = \mu_0 nSI \quad \Lambda' = n\Phi = \mu_0 n^2 SI, L' = \mu_0 n^2 S$

**Eg. Two coils of  $N_1$  and  $N_2$  turns are wound concentrically on a straight cylindrical core of radius  $a$  and permeability  $\mu$ . The windings have lengths  $l_1$  and  $l_2$ , respectively. Find the mutual inductance between the coils.**

(Sol.)  $\Phi_{12} = \mu\left(\frac{N_1}{l_1}\right)(\pi a^2)I_1, \Lambda_{12} = N_2\Phi_{12} = \frac{\mu}{l_1} N_1 N_2 \pi a^2 I_1$

$$\Rightarrow L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\mu}{l_1} N_1 N_2 \pi a^2$$

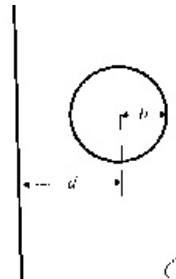
**Eg. Determine the mutual inductance between a very long, straight wire and a conducting circular loop. [台大電研、清大物理所]**

(Sol.)

$B$  at  $p$  is  $\frac{\mu_0 I}{2\pi(d + r \cos \theta)}$

$$\Lambda = \frac{\mu_0 I}{2\pi} \int_0^b \int_0^{2\pi} \frac{rd\theta dr}{d + r \cos \theta} = \frac{\mu_0 I}{2\pi} \int_0^b \frac{2\pi r dr}{\sqrt{d^2 - r^2}} = \mu_0 I (d - \sqrt{d^2 - b^2})$$

$$L = \mu_0 (d - \sqrt{d^2 - b^2})$$



**Eg. Determine the mutual inductance between a conducting triangular loop and a very long straight wire.**

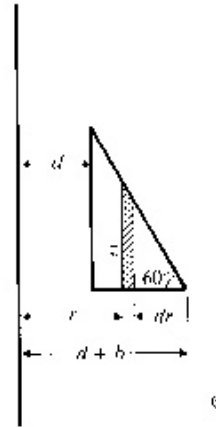
(Sol.)

$$\vec{B}_2 = \hat{a}_\phi \frac{\mu_0 I_2}{2\pi r}, \quad \Lambda = \Phi = \int_S \vec{B} \cdot d\vec{S}, \quad \text{where } d\vec{S} = \hat{a}_\phi z dr$$

$$z = \sqrt{3}(d + b - r)$$

$$\Lambda = \frac{\sqrt{3}\mu_0 I}{2\pi} \int_d^{d+b} \frac{1}{r} (d + b - r) dr = \frac{\sqrt{3}\mu_0 I}{2\pi} [(d + b) \ln(1 + b/d) - b]$$

$$L = \frac{\Lambda}{I} = \frac{\sqrt{3}\mu_0}{2\pi} [(d + b) \ln(1 + b/d) - b]$$



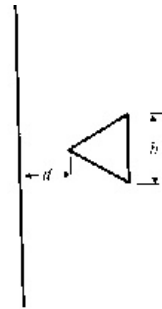
**Eg. Determine the mutual inductance between a very long, straight wire and a conducting equilateral triangular loop. [高考]**

(Sol.)

$$\vec{B} = \hat{a}_\phi \frac{\mu_0 I}{2\pi r} = \hat{a}_\phi B_\phi$$

$$\Lambda = \int_d^{d+\frac{\sqrt{3}b}{2}} B_\phi \cdot \frac{2}{\sqrt{3}} (r - d) dr = \frac{\mu_0 I}{\sqrt{3}\pi} \left[ \frac{\sqrt{3}}{2} h - d \ln\left(1 + \frac{\sqrt{3}h}{2d}\right) \right]$$

$$L = \frac{\Lambda}{I} = \frac{\mu_0}{\sqrt{3}\pi} \left[ \frac{\sqrt{3}}{2} h - d \ln\left(1 + \frac{\sqrt{3}h}{2d}\right) \right]$$

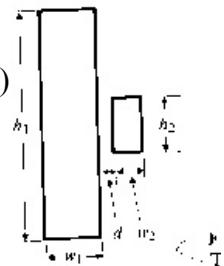


**Eg. Find the mutual inductance between two coplanar rectangular loops with parallel sides. Assume that  $h_1 \gg h_2$  ( $h_2 > w_2 > d$ ). (台大電研)**

(Sol.)

$$\Lambda_{12} = \frac{\mu_0 h_2 I}{2\pi} \int_0^{w_2} \left( \frac{1}{d+x} - \frac{1}{w_1+d+x} \right) dx = \frac{\mu_0 h_2 I}{2\pi} \ln\left( \frac{w_2+d}{d} \cdot \frac{w_1+d}{w_1+w_2+d} \right)$$

$$L_{12} = \frac{\Lambda_{12}}{I} = \frac{\mu_0 h_2}{2\pi} \ln\left[ \frac{(w_1+d)(w_2+d)}{d(w_1+w_2+d)} \right]$$



**Neumaun formula:** 
$$L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C1} \oint_{C2} \frac{d\vec{\ell}_1 d\vec{\ell}_2}{R}$$

$$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \iint_S \vec{B}_1 \cdot d\vec{S}_2 = \frac{N_2}{I_1} \iint_{S_2} (\nabla \times \vec{A}_1) \cdot d\vec{S}_2 = \frac{N_2}{I_1} \oint_{C_2} \vec{A}_1 \cdot d\vec{\ell}_2$$

$$\therefore \vec{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1} \frac{d\vec{\ell}_1}{R_1}, \quad \therefore L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 d\vec{\ell}_2}{R}$$

#### 4-10 Magnetic Energy

$$W_m = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N L_{jk} I_j I_k = \frac{1}{2} \sum_{k=1}^N I_k \Phi_k = \frac{1}{2} \iiint_{V'} \vec{A} \cdot \vec{J} dv'$$

Let  $V_1 = L_1 \frac{di_1}{dt} \Rightarrow W_1 = \int V_1 i_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2 = \frac{1}{2} I_1 \Phi_1$  : Magnetic energy

Similarly,  $V_{21} = L_{21} \frac{di_2}{dt} \Rightarrow W_{21} = \int V_{21} I_1 dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2$

And  $W_2 = \frac{1}{2} L_2 I_2^2 \Rightarrow W_m = \frac{1}{2} L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2} L_2 I_2^2 = \frac{1}{2} \sum_{j=1}^2 \sum_{k=1}^2 L_{jk} I_j I_k$

Generally,  $W_m = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N L_{jk} I_j I_k = \frac{1}{2} \sum_{k=1}^N I_k \Phi_k$  when  $\Phi_k = \sum_{j=1}^N L_{jk} I_j$

$$\therefore \Phi_k = \iint_{S_k} \vec{B} \cdot d\vec{S}_k = \oint_{C_k} \vec{A} \cdot d\vec{\ell}_k$$

$$\therefore W_m = \frac{1}{2} \sum_{k=1}^N \Delta I_k \oint_{C_k} \vec{A} \cdot d\vec{\ell}_k = \frac{1}{2} \iiint_{V'} \vec{A} \cdot \vec{J} dv' \quad (\Delta I_k dl_k' = J(\Delta \hat{a}_k') dl_k' = J \cdot \nu_k')$$

$$\therefore \nabla \cdot (\vec{A} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{H}) \Rightarrow \vec{A} \cdot (\nabla \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{A}) - \nabla \cdot (\vec{A} \times \vec{H})$$

And  $\vec{J} = \nabla \times \vec{H} \Rightarrow \vec{A} \cdot \vec{J} = \vec{H} \cdot \vec{B} - \nabla \cdot (\vec{A} \times \vec{H})$

$$\Rightarrow W_m = \frac{1}{2} \iiint_{V'} \vec{H} \cdot \vec{B} dv' - \frac{1}{2} \iint_{S'} (\vec{A} \times \vec{H}) \cdot \vec{a}_n dS' \quad \text{as} \quad R \rightarrow \infty \Rightarrow |\vec{A}| \propto \frac{1}{R}, \quad |\vec{H}| \propto \frac{1}{R^2},$$

$$d\vec{S} \propto R^2 \Rightarrow -\frac{1}{2} \iint_{S'} (\vec{A} \times \vec{H}) \cdot \hat{a}_n dS' \rightarrow 0$$

$$\therefore W_m = \frac{1}{2} \iiint_{V'} \vec{H} \cdot \vec{B} dv' = \iiint_{V'} w_m dv'$$

$$w_m = \frac{1}{2} \vec{H} \cdot \vec{B}$$

**Magnetic energy density:**  $w_m = \frac{|B|^2}{2\mu}$  and  $L = \frac{2W_m}{I^2}$ .

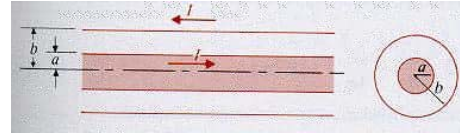
$$w_m = \frac{1}{2} \mu |H|^2$$

**Eg. Determine the inductance per unit length of an air coaxial transmission line that has a solid inner conductor of radius  $a$  and a very thin outer conductor of radius  $b$ . [台科大電機所]**

(Sol.)

$$W_{m1} = \frac{1}{2\mu_0} \int_0^a B_1^2 2\pi r dr = \frac{\mu_0 I^2}{4\pi a^4} \int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi}$$

$$W_{m2} = \frac{1}{2\mu_0} \int_a^b B_2^2 2\pi r dr = \frac{\mu_0 I^2}{4\pi} \int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a}, L' = \frac{2}{I^2} (W_{m1} + W_{m2}) = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi} \ln \frac{b}{a}$$



**Eg. Consider two coupled circuits having self-inductance  $L_1$  and  $L_2$ , which carry currents  $I_1$  and  $I_2$ , respectively. The mutual inductance between the circuits is  $M$ .**

**a) Find the ratio  $I_1/I_2$  that makes the stored magnetic energy  $W_m$  a minimum.**

**b) Show that  $M \leq \sqrt{L_1 L_2}$ . [清大核工所]**

(Sol.)  $W_m = \frac{1}{2} L_1 I_1^2 + M I_1 I_2 + \frac{1}{2} L_2 I_2^2$

(a)  $W_m = \frac{I_2^2}{2} [L_1 (\frac{I_1}{I_2})^2 + 2M (\frac{I_1}{I_2}) + L_2] = \frac{I_2^2}{2} [L_1 x^2 + 2Mx + L_2], x \equiv \frac{I_1}{I_2}$

$$\frac{dW_m}{dx} = 0 = \frac{I_2^2}{2} (2L_1 x + 2M) \Rightarrow x = \frac{I_1}{I_2} = -\frac{M}{L_1} \text{ for minimum } W_m$$

(b)  $(W_m)_{\min} = \frac{I_2^2}{2} (-\frac{M^2}{L_1} + L_2) \geq 0 \Rightarrow M \leq \sqrt{L_1 L_2}$

## 4-11 Magnetic Forces and Torques

### Force due to constant flux linkage:

$$\vec{F}_\phi \cdot d\vec{\ell} = -dW_m = -(\nabla W_m) \cdot d\vec{\ell} \Rightarrow \vec{F}_\phi = -\nabla W_m \text{ and } (T_\phi)_z = -\frac{\partial W_m}{\partial \phi}$$

### Force due to constant current:

$$dW_s = \sum_k I_k d\Phi_k = dW + dW_m$$

$$dW_m = \frac{1}{2} \sum_k I_k d\Phi_k = \frac{1}{2} dW_s \Rightarrow dW = \vec{F}_I \cdot d\vec{\ell} = dW_m = (\nabla W_m) \cdot d\vec{\ell} \Rightarrow \vec{F}_I = \nabla W_m$$

### Torque in terms of mutual inductance:

$$W_m = \frac{1}{2} L_1 I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2 \Rightarrow F_I = I_1 I_2 (\nabla L_{12}), T_I = I_1 I_2 \frac{\partial L_{12}}{\partial \phi}$$

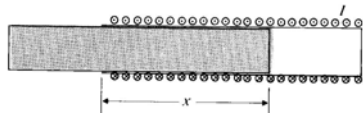


**Eg. One end of a long air-core coaxial transmission line having an inner conductor of radius  $a$  and an outer conductor of inner radius  $b$  is short-circuited by a thin, tight-fitting conducting washer. Find the magnitude and the direction of the magnetic force on the washer when a current  $I$  flows in the line.**

(Sol.)  $W_m = \frac{1}{2} LI^2$ ,  $L = \frac{\Phi}{I} = \frac{x}{I} \int_a^b B_\phi dr = \frac{x}{I} \int_a^b \frac{\mu_0 I}{2\pi r} dr = \frac{\mu_0 x}{2\pi} \ln \frac{b}{a}$

$F_I = \hat{x} \frac{\partial W_m}{\partial x} = \hat{x} \left( \frac{I^2}{2} \right) \frac{\partial L}{\partial x} = \hat{x} \frac{\mu_0 I^2}{4\pi} \ln \left( \frac{b}{a} \right)$

**Eg. A current  $I$  flows in a long solenoid with  $n$  closely wound coil-turns per unit length. The cross-sectional area of its iron core, which has permeability  $\mu$ , is  $S$ . Determine the force acting on the core if it is withdrawn to the position. [高考電**



機技師]

(Sol.)

$W_m = \frac{1}{2} \iiint \mu H^2 dv$ ,  $W_m(x + \Delta x) - W_m(x) = \left( \frac{1}{2} \mu_0 \mu_r n^2 I^2 - \frac{1}{2} \mu_0 n^2 I^2 \right) S \Delta x = \frac{1}{2} \mu_0 (\mu_r - 1) n^2 I^2 S \Delta x$

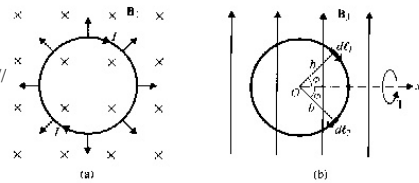
$\Rightarrow (F_I)_x = \frac{\partial W_m}{\partial x} = \frac{\mu_0}{2} (\mu_r - 1) n^2 I^2 S$

**Magnetic torque:**  $\vec{T} = \vec{m} \times \vec{B}$  ( $\vec{B} = \vec{B}_\perp + \vec{B}_\parallel$ ,  $\vec{m} \parallel \vec{B}_\perp \Rightarrow \vec{m} \times \vec{B}_\perp = 0$ )

$d\vec{T} = \hat{x} dF 2b \sin \phi = \hat{x} (I dl B_\parallel \sin \phi) 2b \sin \phi = \hat{x} 2I b^2 B_\parallel \sin^2 \phi d\phi$

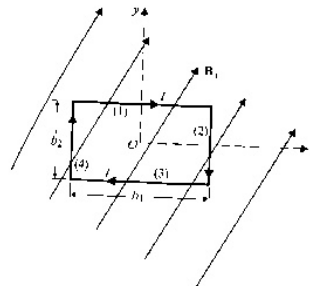
$\vec{T} = \int d\vec{T} = \hat{x} 2I b^2 B_\parallel \int_0^\pi \sin^2 \phi d\phi = \hat{x} I (\pi b^2) B_\parallel = \hat{x} m B_\parallel$

$\Rightarrow \vec{T} = \vec{m} \times \vec{B}$



**Eg. A rectangular loop in the  $xy$ -plane with sides  $b_1$  and  $b_2$  carrying a current  $I$  has in a uniform magnetic field  $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$ . Determine the force and torque on the loop.**

(Sol.)  $\vec{T} = \vec{m} \times \vec{B} = I b_1 b_2 (\hat{x} B_y - \hat{y} B_x)$



## 4-12 Magnetic Circuits

Define  $V_m = NI$  : mmf,  $\Phi = BS$  : magnetic flux,  $\mathfrak{R} = \frac{l}{\mu S}$  : reluctance

$$(1) \sum N_j I_j = \sum_k \mathfrak{R}_k \Phi_k \quad (2) \because \nabla \cdot \vec{B} = 0, \therefore \sum_j \Phi_j = 0$$

**Ex. (a)** Steady current  $I_1$  and  $I_2$  flow in windings of  $N_1$  and  $N_2$  turns, respectively, on the outside legs of the ferromagnetic core. The core has a cross-sectional area  $S_c$  and permeability  $\mu$ . Determine the magnetic flux in the center leg.

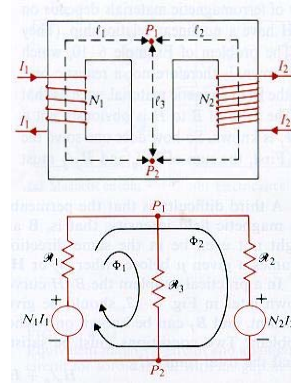
(Sol.)

$$\mathfrak{R}_1 = \frac{l_1}{\mu S_c}, \quad \mathfrak{R}_2 = \frac{l_2}{\mu S_c}, \quad \mathfrak{R}_3 = \frac{l_3}{\mu S_c}$$

$$\text{Loop 1: } N_1 I_1 = (\mathfrak{R}_1 + \mathfrak{R}_3) \Phi_1 + \mathfrak{R}_1 \Phi_2$$

$$\text{Loop 2: } N_1 I_1 - N_2 I_2 = \mathfrak{R}_1 \Phi_1 + (\mathfrak{R}_1 + \mathfrak{R}_2) \Phi_2$$

$$\Phi_1 = \frac{\mathfrak{R}_2 N_1 I_1 + \mathfrak{R}_1 N_2 I_2}{\mathfrak{R}_1 \mathfrak{R}_2 + \mathfrak{R}_1 \mathfrak{R}_3 + \mathfrak{R}_2 \mathfrak{R}_3}$$



**Ex. A** toroidal iron core of relative permeability  $3000\mu_0$  has a mean radius  $R=80\text{mm}$  and a circular cross section with radius  $b=25\text{mm}$ . An air gap  $l_g=3\text{mm}$  exists, and a current  $I$  flows in a 500-turn winding to produce a magnetic flux of  $10^{-5}\text{Wb}$ . Neglecting flux leakage and using mean path length, find (a) the reluctances of the air gap and of the iron core, (b)  $B_g$  and  $H_g$  in the air gap, and  $B_c$  and  $H_c$  in the iron core, (c) the required current  $I$ .

$$\text{(Sol.) (a) } \mathfrak{R}_g = \frac{l_g}{\mu_0 S} = \frac{3 \times 10^{-3}}{4\pi \times 10^{-7} \times (\pi \times 0.025^2)} = 1.21 \times 10^6 \quad (H^{-1})$$

$$\mathfrak{R}_c = \frac{2\pi \cdot 0.08 - 0.003}{3000 \times (4\pi \times 10^{-7}) \times (\pi \times 0.025^2)} = 6.75 \times 10^4 \quad (H^{-1})$$

$$\text{(b) } \vec{B}_g = \vec{B}_c = \hat{a}_\phi \frac{10^{-5}}{\pi \times 0.025^2} = \hat{a}_\phi 5.09 \times 10^{-3} \text{ (T)} \quad H_g = \frac{B_g}{\mu_0}, \quad H_c = \frac{B_c}{3000\mu_0}$$

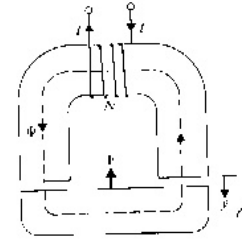
$$\text{(c) } NI = \Phi(\mathfrak{R}_c + \mathfrak{R}_g) \Rightarrow I = 0.0256 \text{ (A)}$$

**Eg. Consider the electromagnet in Figure. In which a current  $I$  in an  $N$ -turn coil produce a flux  $\Phi$  in the magnetic circuit. The cross-sectional area of the core is  $S$ . Determine the lifting force on the armature.**

(Sol.)

$$dW_m = d(W_m)_{airgap} = 2\left(\frac{B^2}{2\mu_0} S dy\right) = \frac{\Phi^2}{\mu_0 S} dy$$

$$L = \frac{N\Phi}{I} = \frac{N \cdot \frac{NI}{R_c + \frac{2y}{\mu_0 S}}}{I}$$



$$F_\phi = \hat{y} \frac{-dW_m}{dy} = -\hat{y} \frac{\Phi^2}{\mu_0 S} \quad \text{and} \quad F_l = \hat{y} \frac{d}{dy} \left( \frac{1}{2} LI^2 \right) = -\hat{y} \frac{1}{\mu_0 S} \left( \frac{NI}{R_c + 2y/\mu_0 S} \right)^2 = -\hat{y} \frac{\Phi^2}{\mu_0 S}$$